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Center for Exact and Earth Sciences
Department of Informatics and Applied Mathematics
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Algebraization in quasi-Nelson logics

Clodomir Silva Lima Neto

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Clodomir Silva Lima Neto

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Advisor

Umberto Rivieccio

Co-advisor

João Marcos

PPgSC – Postgraduate Program in Systems and Computing

DIMAp – Department of Informatics and Applied Mathematics

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Prof. Dr. Umberto Rivieccio
Advisor
Department of Informatics and Applied Mathematics
Federal University of Rio Grande do Norte

Prof. Dr. João Marcos
Co-advisor
Department of Informatics and Applied Mathematics
Federal University of Rio Grande do Norte

Prof. Dr. Regivan Hugo Nunes Santiago
Department of Informatics and Applied Mathematics
Federal University of Rio Grande do Norte

Prof. Dr. Rodolfo Cristian Ertola Biraben
Center for Logic, Epistemology and History of Science
State University of Campinas

Natal-RN, October 31, 2023.

I dedicate this work to my uncle Antônio Pádua Silva (in memoriam).

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What we know is a drop; what we ignore is an ocean.

Sir Isaac Newton.

Algebrização em lógicas quase-Nelson

Autor: Clodomir Silva Lima Neto

Orientador: Umberto Rivieccio

Resumo

A lógica quase-Nelson é uma generalização recentemente introduzida da lógica construtiva com negação forte de Nelson para um cenário não involutivo. O presente trabalho se propõe a estudar a lógica de alguns fragmentos da lógica de quase-Nelson, a saber: pocrims ($\mathcal{L}_{\mathbf{QNP}}$) e semihoops ($\mathcal{L}_{\mathbf{QNS}}$); além da lógica de quase-N4-reticulados ($\mathcal{L}_{\mathbf{QN4}}$). Isso é feito por meio de uma axiomatização através de um cálculo finito no estilo Hilbert. A principal questão que abordaremos é se a semântica algébrica de um determinado fragmento da lógica quase-Nelson (ou classe quase-N4-reticulados) pode ser axiomatizada por meio de equações ou quase-equações. A ferramenta matemática utilizada nesta investigação será a representação twist-álgebra. Chegando à questão da algebrização, lembramos que a lógica quase-Nelson (como extensão de $\mathcal{FL}_{ew}$) é algebrizável no sentido de Blok e Pigozzi. Além disso, mostramos a algebrizabilidade de $\mathcal{L}_{\mathbf{QNP}}$, $\mathcal{L}_{\mathbf{QNS}}$ e $\mathbf{L}_{\mathbf{QN4}}$, que é BP-algebrizável com o conjunto de equações definidoras $E(x) := \{x = x \to x\}$ e o conjunto de fórmulas de equivalência $\Delta(x, y) := \{x \to y, y \to x, \sim x \to \sim y, \sim y \to \sim x\}$.

Palavras-chave: Lógica quase-Nelson. Quase-N4-reticulados. Lógica Algebrizável. Estruturas Twist.

Algebraization in quasi-Nelson logics

Author: Clodomir Silva Lima Neto

Advisor: Umberto Rivieccio

Abstract

Quasi-Nelson logic is a recently introduced generalization of Nelson's constructive logic with strong negation to a non-involutive setting. The present work proposes to study the logic of some fragments of quasi-Nelson logic, namely: pocrims ($\mathcal{L}_{\mathbf{QNP}}$) and semihoops ($\mathcal{L}_{\mathbf{QNS}}$); in addition to the logic of quasi-N4-lattices ($\mathcal{L}_{\mathbf{QN4}}$). This is done by means of an axiomatization via a finite Hilbert-style calculus. The principal question which we will address is whether the algebraic semantics of a given fragment of quasi-Nelson logic (or class of quasi-N4-lattices) can be axiomatized by means of equations or quasi-equations. The mathematical tool used in this investigation will be the twist-algebra representation. Coming to the question of algebraizability, we recall that quasi-Nelson logic (as extensions of $\mathcal{FL}_{ew}$) is algebraizable in the sense of Blok and Pigozzi. Furthermore, we showed the algebraizability of $\mathcal{L}_{\mathbf{QNP}}$, $\mathcal{L}_{\mathbf{QNS}}$ and $\mathcal{L}_{\mathbf{QN4}}$, which is BP-algebraizable with the set of defining equations $E(x) := \{x = x \to x\}$ and the set of equivalence formulas $\Delta(x, y) := \{x \to y, y \to x, \sim x \to \sim y, \sim y \to \sim x\}$.

Keywords: Quasi-Nelson logic. Quasi-N4-lattices. Algebraizable logic. Twist-structures.

Contents

1 Introduction

In this chapter we present the objective of our research and, informally, what it means to algebraize a logic. This dissertation aims to present algebraic semantics for the logic of quasi-N4-lattices and for two fragments of quasi-Nelson logic.

Constructive logic with strong negation ($\mathcal{N}3$) introduced by David Nelson in [18] is a conservative expansion of positive intuitionist logic with an involutive negation. Nelson's paraconsistent logic ($\mathcal{N}4$), a generalization of $\mathcal{N}3$ obtained by abandoning the explosive axiom $p \to ({\sim} p \to q)$, appears later in a paper together with Ahmad Almukdad [1]. $\mathcal{N}3$ and $\mathcal{N}4$ have, as algebraic semantics, the variety of Nelson algebras and the variety of N4-lattices, respectively.

Another generalization of $\mathcal{N}3$ is obtained by abandoning the double negation axiom ${\sim}{\sim} p \to p$. This is quasi-Nelson logic ($\mathcal{QNL}$), which was introduced in [25] and whose algebraic semantics is the variety of quasi-Nelson algebras. Recent research ([20], [21], [24], [17]) has focused on the question of characterizing logics/algebras that correspond to fragments of $\mathcal{QNL}$.

Umberto Rivieccio [22] introduced the class of quasi-N4-lattices (QN4-lattices), as a common generalization of the varieties of N4-lattices and the varieties of quasi-Nelson algebras. In other words, N4-lattices are precisely the quasi-N4-lattices satisfying the law of double negation, and quasi-Nelson algebras are precisely the QN4-lattices satisfying the explosive law.

In most general terms, we may say that algebraizing a logic consists in obtaining a class of algebras whose equational consequence mirrors the behavior of logic. Thus, the goal of algebraization is to obtain a relation in which the elements of an algebra represent "generalized truth values" of the logic, the connectives of the logic are correspond to algebraic operations and the axioms of a logic are interpreted as equations valid in algebra. In the case of the most well-behaved logics, it may be shown that the logical consequence and the equational consequence relation are equivalent in a strong sense. A well-known example of this relationship is the one Boolean algebras to classical propositional logic.

We understand by Abstract Algebraic Logic the set of techniques, results and studies on this relationship, between the logics and respective algebras. Coming to the question of algebraizability, we recall that both $\mathcal{N}3$ and $\mathcal{QNL}$ (as extensions of $\mathcal{FL}_{ew}$ - Full Lambek calculus with exchange and weakening) are algebraizable in the sense of Blok and Pigozzi [2]; for more details, see [19] and [14].

In this dissertation, we propose an algebraization for $\mathcal{L}_{\mathbf{QN4}}$, for $\mathcal{L}_{\mathbf{QNP}}$ and for $\mathcal{L}_{\mathbf{QNS}}$ by the method of Blok and Pigozzi. Actually, the main result is to introduce logics and show that they are algebraizable with respect to classes of algebras that had been interpreted in the papers by Umberto Rivieccio and to characterize the corresponding fragments of the logics, which until now had not been presented.

The present document is organized as follows: Chapter 2 introduces the basic concepts and terminology involving algebra, logics and their algebraization. Chapter 3 introduces the concept of nuclei. Then, chapters 4 and 5 describes our proposal for the algebraization of non-involutive Nelson logics, in 4 we presents of quasi-N4-lattices and their logic $\mathcal{L}_{\mathbf{QN4}}$; in 5 we present some fragments of $\mathcal{QNL}$, in particular, quasi-Nelson pocrims and their logic $\mathcal{L}_{\mathbf{QNP}}$; and quasi-Nelson semihoops and their logic $\mathcal{L}_{\mathbf{QNS}}$. In the conclusion, we reflect upon the results obtained and indicate some directions for future developments.

2 Theoretical Background

In this chapter we introduce the theory and methods that will be used in this document. We present two components to specify a logic, namely: a *language* (the "formulas" of the logic), and a *relation of consequence* (derivability, inference), often denoted by $\vdash$. This relation can be defined or presented in several ways, here we follow the deductive way: we use some concepts of proof in a formal system, normally called *calculus*. The two kinds main are the so-called *Hilbert-style* or axiomatic calculi, and *Gentzen-style* or sequent calculi, here we use the first kind. At the end of this chapter, we introduce the process of *algebraization*, that is, the process by which we associate a certain class of algebras to a particular deductive system (or logic).

2.1 Algebra

In this section, we present some definitions of basic elements of the study of Universal Algebra, whose history is strongly linked to the study of the relationship between Logic and Mathematics. For a more complete presentation, we recommend [6].

Definition 1 ([6], Def. 1.1, Cha. I)**.** A nonempty set L together with two binary operations $\wedge$ and $\vee$ (read "meet" and "join" respectively) on L is called a **lattice** if it satisfies the following equations:

(L1) commutative laws: $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.

(L2) associative laws: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$.

(L3) idempotent laws: $x \wedge x = x$ and $x \vee x = x$.

(L4) absorption laws: $x = x \wedge (x \vee y)$ and $x = x \vee (x \wedge y)$.

Before introducing the second definition of a lattice we need the notion of a partial order on a set.

Definition 2 ([6], Def. 1.2, Cha. I)**.** A binary relation $\leq$ defined on a set A is a **partial order** on the set A if the following conditions hold identically in A:

1. reflexivity: $a \leq a$.
2. antisymmetry: $a \leq b$ and $b \leq a$ imply $a = b$
3. transitivity: $a \leq b$ and $b \leq c$ imply $a \leq c$.

A nonempty set with a partial order on it is called a partially ordered set or **poset**.

Remark 1. A relation $\leq$ on a set A which is reflexive and transitive but not necessarily antisymmetric is called **quasiorder** or **pre-order**.

Example 1 ([6], Exa. 1, Cha. I)**.** Let $\wp(A)$ denote the power set of A, i.e. the set of all subsets of A. Then $\subseteq$ is a partial order on $\wp(A)$.

Definition 3 ([6], Def. 1.3, Cha. I)**.** Let A be a subset of a poset P. An element $p \in P$ is an **upper bound** for A if $a \leq p$ for every $a \in A$. An element $p \in P$ is the least upper bound of A, or **supremum** of A ($\sup A$) if p is an upper bound of A, and $a \leq b$ for every $a \in A$ implies $p \leq b$. Dually we can define what it means for p to be a **lower bound** of A, and for p to be the greatest lower bound of A, also called the **infimum** of A ($\inf A$).

Now let us look at the second approach to lattices.

Definition 4 ([6], Def. 1.4, Cha. I)**.** A poset L is a **lattice** iff for every $a, b \in L$ both $\sup\{a, b\}$ and $\inf\{a, b\}$ exist (in L).

Definition 5 ([6], Def. 3.1, Cha. I)**.** A **distributive lattice** is a lattice which satisfies the distributive laws:

(DL1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

(DL2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Theorem 1 ([6], Thm. 3.2, Cha. I)**.** A lattice L satisfies **(DL1)** iff it satisfies **(DL2)**.

We have now established the concept of an algebra; focusing on this, we discuss the notions of subalgebra, congruence, quotient algebra, homomorphism, direct product, variety, term algebras and free algebra.

Definition 6 ([6], Def. 1.1, Cha. II)**.** For A a nonempty set and n a nonnegative integer we define $A^0 = \{\varnothing\}$ and for $n > 0$, A^n is the set of n-tuples of elements from A. An **n-ary operation** (or function) on A is any function f from A^n to A; n is the arity (or rank) of f. A finitary operation is an n-ary operation, for some n. The image of $\langle a_1, \ldots, a_n \rangle$ under an n-ary operation f is denoted by $f(a_1, \ldots, a_n)$. An operation f on A is called a nullary operation (or constant) if its arity is zero; it is completely determined by the image $f(\varnothing)$ in A of the only element $\varnothing$ in A^0. Thus a nullary operation is thought of as an element of A. An operation f on A is unary, binary or ternary if its arity is 1, 2, or 3, respectively.

Definition 7 ([6], Def. 1.2, Cha. II)**.** A **algebraic language** (or type) of algebras is a set F of function symbols such that a nonnegative integer n is assigned to each member f of F. This integer is called the arity (or rank) of f, and f is said to be an n-ary function symbol. The subset of n-ary function symbols in F is denoted by F_n.

When specifying a particular language, it is customary to describe language and the function f as the sequence; for instance, one says "let $\langle \wedge, \vee, \rightarrow, \bot, \top \rangle$ be a language of type $\langle 2, 2, 2, 0, 0 \rangle$".

Definition 8 ([6], Def. 1.3, Cha. II)**.** If F is a language of algebras then an **algebra** $\mathbf{A}$ of type $\mathcal{F}$ is an ordered pair $\langle A;\ F \rangle$ where A is a nonempty set called universe of $\mathbf{A}$; and F is a family of finitary operations on A indexed by the language F such that corresponding to each n-ary function symbol f in $\mathcal{F}$ there is an n-ary operation $f^{\mathbf{A}}$ on A, where $f^{\mathbf{A}}$'s are called the fundamental operations of $\mathbf{A}$.

When an algebraic language F is interpreted in a domain or mathematical universe, to specify one of them one writes $\mathbf{A} = \langle A;\ f(a_1, \ldots, a_n)\rangle$. It is traditional to write 0 for $\perp^{\mathbf{A}}$ and 1 for $\top^{\mathbf{A}}$. For instance, one says "let $\mathbf{A} = \langle A;\ \to, \sim, 0, 1\rangle$ be an algebra of type $\langle 2, 1, 0, 0\rangle$".

Example 2 ([6], Exa. 1, Cha. II)**.** A **group** is an algebra $\mathbf{G} = \langle G;\ *, ^{-1}, 1\rangle$ of type $\langle 2, 1, 0\rangle$ in which the following equations are true:

(G1) $x * (y * z) = (x * y) * z$.

(G2) $x * 1 = 1 * x = x$.

(G3) $x * x^{-1} = x^{-1} * x = 1$.

A group $\mathbf{G}$ is **Abelian** (or commutative) if the following equation is true:

(G4) $x * y = y * x$.

Example 3 ([6], Exa. 2, Cha. II)**.** A **semigroup** is a ordered pair $\langle S;\ *\rangle$ in which **(G1)** is true. A **monoid** is an algebra $\mathbf{M} = \langle M;\ *, 1\rangle$ of type $\langle 2, 0\rangle$ satisfying **(G1)** and **(G2)**.

Example 4 ([6], Exa. 7, Cha. II)**.** A **semilattice** is a semigroup $\langle S;\ *\rangle$ which satisfies the commutative law **(G4)** and the idempotent law

(S1) $x * x = x$.

Example 5 ([6], Exa. 8, Cha. II)**.** A **lattice** is an algebra $\mathbf{L} = \langle L;\ \wedge, \vee\rangle$ of type $\langle 2, 2\rangle$ which satisfies **(L1)**–**(L4)**.

Example 6 ([6], Exa. 9, Cha. II)**.** A **bounded lattice** is an algebra $\mathbf{A} = \langle A;\ \wedge, \vee, 0, 1\rangle$ of type $\langle 2, 2, 0, 0\rangle$ which satisfies:

(BL1) $\langle A;\ \wedge, \vee\rangle$ is a lattice.

(BL2) $x \wedge 0 = 0$ and $x \vee 1 = 1$.

Example 7 ([15], Def. 1)**.** A **Brouwerian algebra** or **implicative lattice** is an algebra $\mathbf{B} = \langle B;\ \wedge, \vee, \to\rangle$ of type $\langle 2, 2, 2\rangle$ such that:

(B1) $\langle B;\ \wedge, \vee\rangle$ is a lattice with order $\leq$.

(B2) For a, b, $c \in B$, $a \wedge b \leq c$ iff $a \leq b \to c$.

Example 8 ([23], Def. 2.1)**.** An algebra $\mathbf{H} = \langle H;\ \to, 1\rangle$ of type $\langle 2, 0\rangle$ is called **Hilbert algebra** if the following hold.

(H1) $x \to (y \to x) = 1$.

(H2) $x \to (y \to z) = (x \to y) \to (x \to z)$.

(H3) if $x \to y = y \to x = 1$ then $x = y$.

Example 9 ([6], Exa. 11, Cha. II)**.** An algebra $\mathbf{H} = \langle H;\ \wedge, \vee, \to, 0, 1\rangle$ of type $\langle 2, 2, 2, 0\rangle$ is called **Heyting algebra** if the following hold.

(HA1) $\langle H;\ \wedge, \vee\rangle$ is a distributive lattice.

(HA2) $x \wedge 0 = 0$ and $x \vee 1 = 1$.

(HA3) $x \to x = 1$.

(HA4) $(x \to y) \wedge y = y$ and $x \wedge (x \to y) = x \wedge y$.

(HA5) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$ and $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

Definition 9 ([6], Exa. 10, Cha. II)**.** An algebra $\mathbf{B} = \langle B;\ \wedge, \vee, \sim, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ is called **Boolean algebra** if the following hold.

(BA1) $\langle B;\ \wedge, \vee\rangle$ is a distributive lattice.

(BA2) $x \wedge 0 = 0$ and $x \vee 1 = 1$.

(BA3) $x \wedge \sim x = 0$ and $x \vee \sim x = 1$.

Example 10 ([26])**.** A **De Morgan algebra** is an algebra $\mathbf{A} = \langle A;\ \wedge, \vee, \sim, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ which satisfies:

(DM1) $\langle A;\ \wedge, \vee\rangle$ is a distributive lattice.

(DM2) $\sim(x \wedge y) = \sim x \vee \sim y$.

(DM3) $\sim(x \vee y) = \sim x \wedge \sim y$.

(DM4) $x \wedge 0 = 0$.

(DM5) $\sim 1 = 0$.

Example 11 ([16], Def. 1.2)**.** An algebra $\mathbf{A} = \langle A;\ \wedge, \vee, \rightarrow, \sim, 1\rangle$ of type $\langle 2, 2, 2, 1, 0\rangle$ is called **Nelson algebra** (or N-lattice) if the following hold.

(N1) $x \vee 1 = 1$.

(N2) $x \wedge (x \vee y) = x$.

(N3) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$.

(N4) $\sim\sim x = x$.

(N5) $\sim(x \wedge y) = \sim x \vee \sim y$.

(N6) $x \wedge \sim x = (x \wedge \sim x) \wedge (y \vee \sim y)$.

(N7) $x \rightarrow x = 1$.

(N8) $x \wedge (x \rightarrow y) = x \wedge (\sim x \vee y)$.

(N9) $(x \wedge y) \rightarrow z = x \rightarrow (y \rightarrow z)$.

(N10) $(x \rightarrow y) \wedge (\sim x \vee y) = \sim x \vee y$.

(N11) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.

Example 12 ([19], Def. 5.1)**.** An algebra $\mathbf{A} = \langle A;\ \wedge, \vee, \rightarrow, \sim\rangle$ is said to be an **N4-lattice** if the following hold.

(N4.1) The reduct $\langle A;\ \wedge, \vee, \sim\rangle$ is a De Morgan algebra and the following equations hold: $\sim(p \vee q) = \sim p \wedge \sim q$ and $\sim\sim p = p$.

(N4.2) The relation $\preceq$, where $a \preceq b$ denotes $(a \rightarrow b) \rightarrow (a \rightarrow b) = a \rightarrow b$, is a preordering on A.

(N4.3) The relation $=$, where $a = b$ if and only if $a \preceq b$ and $b \preceq a$, is a congruence relation with respect to $\vee$, $\wedge$, $\rightarrow$ and the quotient-algebra $\langle A;\ \vee, \wedge, \rightarrow\rangle / =$ is an implicative lattice.

(N4.4) For any a, $b \in A$, $\sim(a \rightarrow b) = a \wedge \sim b$.

(N4.5) For any a, $b \in A$, $a \leq b$ if and only if $a \preceq b$ and $\sim b = \sim a$ where $\leq$ is a lattice ordering on $\mathbf{A}$.

Example 13 ([11])**.** An algebra $\mathbf{A} = \langle A;\ \wedge, \vee, *, /, \backslash, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 2, 0, 0\rangle$ is called **Full Lambek algebra** or FL-algebra if the following hold.

(FL1) $\langle A;\ \wedge, \vee\rangle$ is a lattice.

(FL2) $\langle A;\ *, 1\rangle$ is a monoid.

(FL3) For x, y, $z \in A$, $x * y \leq z$ iff $x \leq z/y$ iff $y \leq x\backslash z$.

(FL4) 0 is an arbitrary element of A.

Example 14 ([25], Def. 2.1)**.** A **commutative integral bounded residuated lattice** (CIBRL) is an algebra $\mathbf{A} = \langle A;\ \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ such that:

(C1) $\langle A;\ \wedge, \vee, 0, 1\rangle$ is a bounded lattice with order $\leq$.

(C2) $\langle A;\ *, 1\rangle$ is a commutative monoid.

(C3) For a, b, $c \in A$, $a * b \leq c$ iff $b \leq a \Rightarrow c$.

Example 15 ([25], Def. 2.3)**.** A **quasi-Nelson residuated lattice** is a CIBRL that satisfies the Nelson identity: $(x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x)) = x \Rightarrow y$.

In the next pages, we discuss the main notions about congruence.

Definition 10 ([6], Def. 4.4, Cha. I)**.** Let A be a set. A **binary relation** R on A is an equivalence relation on A if, for any a, b, c from A, it satisfies:

(E1) reflexivity: aRa.

(E2) symmetry: aRb implies bRa.

(E3) transitivity: aRb and bRc imply aRc.

Remark 2. Let A be a set. Recall that a binary relation R on A is a subset of $A \times A$. If $\langle a, b\rangle \in R$ then we write aRb. Furthemore, $\mathsf{Eqr}(A)$ is a set of all equivalence relations on A.

Definition 11 ([6], Def. 5.1, Cha. II)**.** Let $\mathbf{A}$ be an algebra of type $\mathcal{F}$ and let $\theta \in \mathsf{Eqr}(A)$. Then θ is a **congruence** on $\mathbf{A}$ if θ satisfies the following compatibility property:

(CP) For each n-ary function symbol $f \in \mathcal{F}$ and elements $a_i,\ b_i \in A$, if $a_i \theta b_i$ holds for $1 \leqslant i \leqslant n$ then $f^{\mathbf{A}}(a_1, \ldots, a_n) \theta f^{\mathbf{A}}(b_1, \ldots, b_n)$ holds.

Remark 3. The set of all congruences on an algebra $\mathbf{A}$ is denoted by $\mathsf{Con}\mathbf{A}$.

Definition 12 ([6], Def. 5.2, Cha. II)**.** Let θ be a congruence on an algebra $\mathbf{A}$. Then the **quotient algebra** of $\mathbf{A}$ by θ, written $\mathbf{A}/\theta$, is the algebra whose universe is A/θ and whose operations satisfy $f^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = f^{\mathbf{A}}(b_1, \ldots, b_n)/\theta$ where $a_1, \ldots, a_n \in A$ and f is an n-ary function symbol in $\mathcal{F}$.

Remark 4. Note that quotient algebras of $\mathbf{A}$ are of the same type as $\mathbf{A}$.

There are several important methods of constructing new algebras from given ones. Three of the most fundamental are the formation of subalgebras, homomorphic images, and direct products.

Definition 13 ([6], Def. 6.1, Cha. II)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras of the same type $\mathcal{F}$. A mapping $\alpha : A \to B$ is called a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$ if

$$\alpha f^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{B}}(\alpha a_1, \ldots \alpha a_n)$$

for each n-ary f in $\mathcal{F}$ and each sequence $a_1, \ldots, a_n$ from A. If, in addition, the mapping α is onto then $\mathbf{B}$ is said to be a **homomorphic image** of $\mathbf{A}$, and α is called an epimorphism.

Remark 5. Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras of the same type $\mathcal{F}$, then $\mathsf{Hom}(A, B)$ denotes the set of all homomorphisms from $\mathbf{A}$ to $\mathbf{B}$.

Definition 14 ([6], Def. 2.1, Cha. II)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras of the same type $\mathcal{F}$. The function $\alpha : A \to B$ is an **isomorphism** from $\mathbf{A}$ to $\mathbf{B}$ if α is one-to-one and onto, and for every n-ary $f \in \mathcal{F}$, for $a_1, \ldots, a_n \in A$, we have $\alpha f^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{B}}(\alpha a_1, \ldots, \alpha a_n)$. We say $\mathbf{A}$ is isomorphic to $\mathbf{B}$, written $\mathbf{A} \cong \mathbf{B}$, if there is an isomorphism from $\mathbf{A}$ to $\mathbf{B}$.

Definition 15 ([6], Def. 2.2, Cha. II)**.** Let **A** and **B** be two algebras of the same type $\mathcal{F}$. Then **B** is a **subalgebra** of **A** if $B \subseteq A$ and every fundamental operation of B is the restriction of the corresponding operation of A, i.e., for each function symbol of f, $f^{\mathbf{B}}$ is $f^{\mathbf{A}}$ restricted to B, we write $\mathbf{B} \leqslant \mathbf{A}$.

Definition 16 ([6], Def. 7.1, Cha. II)**.** Let $\mathbf{A_1}$ and $\mathbf{A_2}$ be two algebras of the same type $\mathcal{F}$. Define the **direct product** $\mathbf{A_1} \times \mathbf{A_2}$ to be the algebra whose universe is the set $A_1 \times A_2$ and such that for $f \in \mathcal{F}_n$, $a_i \in A_1$, $a_i' \in A_2$, $1 \leq i \leq n$,

$$f^{\mathbf{A_1} \times \mathbf{A_2}}(\langle a_1, a_1' \rangle, \ldots \langle a_n, a_n' \rangle) = \langle f^{\mathbf{A_1}}(a_1, \ldots, a_n), f^{\mathbf{A_2}}(a_1', \ldots, a_n') \rangle$$

Definition 17 ([6], Def. 7.2, Cha. II)**.** The mapping $\pi_i : A_1 \times A_2 \to A_i, \quad i \in \{1, 2\}$ defined by $\pi_i(\langle a_1, a_2 \rangle) = a_i$ is called the **projection map** on the ith coordinate of $A_1 \times A_2$.

Definition 18 ([6], Def. 6.7, Cha. II)**.** Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a homomorphism. Then the ***kernel*** of α, written $\ker \alpha$, is defined by $\ker \alpha = \{\langle a, b \rangle \in A \times A; \ \ \alpha(a) = \alpha(b)\}$.

Theorem 2 ([6], Thm. 6.8, Cha. II)**.** Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a homomorphism. Then $\ker \alpha$ is a congruence on **A**.

A major theme in universal algebra is the study of classes of algebras ($\mathcal{K}$) of the same type closed under one or more constructions.

Definition 19 ([6], Def. 9.1, Cha. II)**.** We introduce the following operators mapping classes of algebras to classes of algebras (all of the same type):

- $\mathbf{A} \in \mathcal{I}(\mathcal{K})$ iff **A** is isomorphic to some member of $\mathcal{K}$.
- $\mathbf{A} \in \mathcal{S}(\mathcal{K})$ iff **A** is a subalgebra to some member of $\mathcal{K}$.
- $\mathbf{A} \in \mathcal{H}(\mathcal{K})$ iff **A** is a homomorphic image to some member of $\mathcal{K}$.
- $\mathbf{A} \in \mathcal{P}(\mathcal{K})$ iff **A** is a direct product of a nonempty family of algebras in $\mathcal{K}$.

Definition 20 ([6], Def. 9.3, Cha. II)**.** A nonempty class $\mathcal{K}$ of algebras of type $\mathcal{F}$ is called a **variety** if it is closed under homomorphic images, subalgebras, and direct products.

The **algebras of formulas** (**Fm**) is just what in universal algebra is called the **term algebra**, defined later.

Definition 21 ([6], Def. 10.1, CHa. II)**.** Let X be a set of (distinct) objects called **variables**. Let $\mathcal{F}$ be a type of algebras. The set $T(X)$ of terms of type $\mathcal{F}$ over X is the smallest set such that

(i) $X \cup F_0 \subseteq T(X)$.

(ii) If $p_1, \ldots, p_n \in T(X)$ and $f \in F_n$ then the "string" $f(p_1, \ldots, p_n) \in T(X)$.

Example 16 ([6], Exa. 1, Chap. II)**.** Let $\mathcal{F}$ consist of a single binary function symbol $*$, and let $X = \{x, y, z\}$. Then x, y, z, $x * y$, $y * z$, $x * (y * z)$, $(x * y) * z$ are some of the terms over X.

One can, in a natural way, transform the set $T(X)$ into an algebra.

Definition 22 ([6], Def. 10.4, Cha. II)**.** Given $\mathcal{F}$ and X, if $T(X) \neq \varnothing$ then the **term algebra** of type $\mathcal{F}$ over X, written $\mathbf{T}(X)$, has as its universe the set $T(X)$, and the fundamental operations satisfy $f^{\mathbf{T}(X)} : \langle p_1, \ldots, p_n \rangle \mapsto f(p_1, \ldots, p_n)$ for $f \in F_n$ and $p_i \in T(X)$, $1 \leqslant i \leqslant n$.

Definition 23 ([6], Def. 10.5)**.** Let $\mathcal{K}$ be a class of algebras of type $\mathcal{F}$ and let $\mathbf{U}(X)$ be an algebra of type $\mathcal{F}$ which is generated by X. If for every $\mathbf{A} \in \mathcal{K}$ and for every map $\alpha : X \to A$ there is a homomorphism $\beta : \mathbf{U}(X) \to \mathbf{A}$ which extends α, then we say $U(X)$ has the universal mapping property for $\mathcal{K}$ over X, X is called a **set of free generators** of $\mathbf{U}(X)$, and $\mathbf{U}(X)$ is said to be freely generated by X.

The next syntactic objects constructed from formulas are ***equations*** (or identities) and ***quasi-equations*** (or quasi-identities).

Definition 24 ([6], Def. 11.1, Cha. II)**.** An ***equation*** of type $\mathcal{F}$ over X is an expression of the form $p = q$, where $p,\ q \in T(X)$.

- Let $Id(X)$ be the set of equations of type $\mathcal{F}$ over X. An algebra $\mathbf{A}$ of type $\mathcal{F}$ **satisfies** an equation $p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)$ if for every choice of $a_1, \ldots, a_n \in A$, we have $p^{\mathbf{A}}(x_1, \ldots, x_n) = q^{\mathbf{A}}(x_1, \ldots, x_n)$.
- We say that the equation is **true** in $\mathbf{A}$, or holds in $\mathbf{A}$, and write $\mathbf{A} \vDash p = q$.
- If Σ is a set of equations, we say $\mathbf{A}$ **satisfies** Σ, written $A \vDash \Sigma$ if $\mathbf{A} \vDash p = q$ for each $p = q \in \Sigma$.
- A class $\mathcal{K}$ of algebras **satisfies** $p = q$, written $\mathbf{K} \vDash p = q$, if each member of $\mathbf{K}$ satisfies $p = q$.
- We say $\mathcal{K}$ **satisfies** Σ, written $\mathcal{K} \vDash \Sigma$ if $\mathcal{K} \vDash p = q$ for each $p = q \in \Sigma$.

We can reformulate the above definition of satisfaction using the notion of homomorphism.

Lemma 1 ([6], Lem. 11.2, Cha. II)**.** If $\mathcal{K}$ is a class of algebras of type $\mathcal{F}$ and $p = q$ is an equation of type $\mathcal{F}$ over X, then $\mathcal{K} \vDash p = q$ iff for every $\mathbf{A} \in \mathbf{K}$ and for every homomorphism $\alpha : \mathbf{T}(X) \to \mathbf{A}$ we have $\alpha p = \alpha q$.

Lemma 2 ([6], Lem. 11.3, Cha. II)**.** For any class $\mathcal{K}$ of type $\mathcal{F}$, all of the classes $\mathcal{K}$, $\mathcal{I}(\mathcal{K})$, $\mathcal{S}(\mathcal{K})$, $\mathcal{H}(\mathcal{K})$ and $\mathcal{P}(\mathcal{K})$ satisfy the same equations over any set of variables X.

Remark 6. The set of all equations of the language L is denoted by $\mathsf{Eq}(Fm_L)$ or simply by Eq.

Definition 25 ([6], Def. 11.7, Cha. II)**.** Let Σ be a set of equations of type $\mathcal{F}$, and define $\mathcal{M}(\Sigma)$ to be the class of algebras $\mathbf{A}$ satisfying Σ. A class $\mathcal{K}$ of algebras is an **equational class** if there is a set of equations Σ such that $\mathcal{K} = \mathcal{M}(\Sigma)$. In this case we say that $\mathcal{K}$ is defined, or axiomatized, by Σ.

Theorem 3 (Birkhoff)**.** $\mathcal{K}$ is an equational class iff $\mathcal{K}$ is a variety.

Proof. [6], Theorem 11.9. □

Now, let's move on to the definition of the reduce product, which result from a certain combination of the direct product and quotient constructions.

Remark 7. Let I be a set. Recall that a ***filter*** F over I is a set $F \subseteq \wp(I)$ such that: (i) $I \in F$; (ii) if $X,\ Y \in F$ then $X \cap Y \in F$; and (iii) if $X \in F$ and $X \subseteq Y$ then $Y \in F$. Furthermore, if $\wp(I) \neq F$, then F is called a ***proper filter***; and, if F is maximal (that is, for every filter F', $F' \subseteq F$), then F is called an ***ultrafilter***.

Definition 26 ([6], Def. 2.1, Cha. V)**.** Let $(\mathbf{A}_i)_{i\in I}$ be a nonempty indexed family of structures of type $\mathcal{L}$, and suppose F is a proper filter over I. Define the binary relation θ_F on $\prod\limits_{i\in I} A_i$ by

$$\langle a, b\rangle \in \theta_F \qquad \text{iff} \qquad \{i \in I;\ \ a(i) = b(i)\} \in F$$

Lemma 3 ([6], Lem. 2.2, Cha. V)**.** For $(\mathbf{A}_i)_{i\in I}$ and F as above, the relation θ_F is an equivalence relation on $\prod\limits_{i\in I} A_i$. For a fundamental n-ary operation of $\prod\limits_{i\in I} \mathbf{A}_i$ and for $\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle \in \theta_F$ we have $\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)\rangle \in \theta_F$, i.e., θ_F is a congruence for the **algebra part** of $\mathbf{A}$.

Definition 27 ([6], Def. 2.3, Cha. V)**.** Given a nonempty indexed family of structures $(\mathbf{A}_i)_{i\in I}$ of type $\mathcal{L}$ and a proper filter F over I, define the ***reduce product*** P_R, $\prod\limits_{i\in I} \mathbf{A}_i/F$ as follows. Let its universe $\prod\limits_{i\in I} A_i/F$ be the set $\prod\limits_{i\in I} A_i/\theta_F$, and let a/F denote the element a/θ_F. For f an n-ary function symbol and for $a_1, \ldots, a_n \in \prod\limits_{i\in I} A_i$, let

$$f(a_1/F, \ldots, a_n/F) = f(a_1, \ldots, a_n)/F$$

and for r an n-ary relation symbol, let $r(a_1/F, \ldots, a_n/F)$ hold iff

$$\{i \in I;\ \ \mathbf{A}_i \vDash r(a_1(i), \ldots, a_n(i))\} \in F$$

If K is a nonempty class of structures of type $\mathcal{L}$, let $P_R(K)$ denote the class of all reduced products $P_R, \prod_{i \in I} \mathbf{A}_i/F$, where $\mathbf{A}_i \in F$.

Definition 28 ([6], Def. 2.24, Cha. V)**.** A ***quasi-equation*** is an equation or a formula of the form $(p_1 = q_1 \wedge \ldots \wedge p_n = q_n) \rightarrow (p = q)$. A ***quasi-variety*** is a class of algebras closed under isomorphism, subalgebra and reduce product, and containing the one-element algebra.

Theorem 4 ([6], Thm. 2.25, Cha. V)**.** Let $\mathcal{K}$ be a class of algebras. Then the following are equivalent:

(a) $\mathcal{K}$ can be axiomatized by quasi-equations.

(b) $\mathcal{K}$ is a quasi-variety.

2.2 Logic

After having defined formulas, one can consider other mathematical objects constructed from them. The other linguistic objects that will appear in this document are sequents. In the literature there are several kinds of sequents. The most common here will be pairs $\Gamma \vdash \alpha$ where Γ is a finite set of formulas and α is a formula. This notation of sequents will be used to express (Hilbert-style) rules of logic without postulating them of any particular logic; for instance the popular rule of *Modus Ponens* can be described as the sequent $\{\alpha, \alpha \rightarrow \beta\} \vdash \beta$.

Definition 29 ([10], Def. 1.3)**.** A ***substitution*** is an endomorphism $\sigma : \mathbf{Fm} \rightarrow \mathbf{Fm}$. For each $\alpha \in Fm$, $\sigma\alpha$ is a ***substitution instance*** of α. The set of all substitutions is denoted by $\mathtt{End}(\mathbf{Fm}) := \mathtt{Hom}(\mathbf{Fm}, \mathbf{Fm})$.

Definition 30 ([10], Def. 1.5)**.** A **logic** (of type $\mathbf{L}$) is an ordered pair $\mathcal{L} = \langle \mathbf{L}, \vdash_{\mathcal{L}} \rangle$ where $\mathbf{L}$ is an algebraic language and $\vdash_{\mathcal{L}} \subseteq \wp(\mathbf{Fm}) \times \mathbf{Fm}$ is a relation, called ***consequence relation*** of the logic, satisfying the following properties, for all $\Gamma \cup \Delta \cup \{\alpha\} \subseteq \mathbf{Fm}$:

(R) Reflexivity: $\alpha \in \Gamma$ implies $\Gamma \vdash_{\mathcal{L}} \alpha$.

(M) Monotonicity: $(\Gamma \vdash_{\mathcal{L}} \alpha$ and $\Gamma \subseteq \Delta)$ implies $\Delta \vdash_{\mathcal{L}} \alpha$.

(T) Transitivity: $(\Gamma \vdash_{\mathcal{L}} \alpha$ and $\Delta \vdash_{\mathcal{L}} \beta$ for every $\beta \in \Gamma)$ implies $\Delta \vdash_{\mathcal{L}} \alpha$.

(S) Structurality: $\Gamma \vdash_{\mathcal{L}} \alpha$ implies $\sigma\Gamma \vdash_{\mathcal{L}} \sigma\alpha$ for every substitution σ.

Definition 31 ([10], Def. 1.6)**.** A logic $\mathcal{L}$ is **finitary** when the following holds for all $\Gamma \cup \{\alpha\} \subseteq \mathbf{Fm}$:

(F) $\Gamma \vdash_{\mathcal{L}} \alpha \iff \exists \Delta \subseteq \Gamma$, Δ finite, such that $\Delta \vdash_{\mathcal{L}} \alpha$.

Associated with any logic are its various extensions, expansions and fragments. By an **extension** of a logic $\mathcal{L}$ over the language $\mathbf{L}$ we mean any system $\mathcal{L}' = \langle \mathbf{L}, \vdash_{\mathcal{L}'} \rangle$ over the same language such that $\Gamma \vdash_{\mathcal{L}} \alpha$ implies $\Gamma \vdash_{\mathcal{L}'} \alpha$ for all $\Gamma \cup \{\alpha\} \subseteq Fm$; $\mathcal{L}$ is called a **conservative expansion** of $\mathcal{L}'$ in this case. $\mathcal{L}'$ is an **axiomatic extension** of $\mathcal{L}$ if it is obtained by adjoining new axioms but leaving the rules of inference fixed. Let $\mathcal{L}'$ be a sublanguage of $\mathcal{L}$, and let $\vdash_{\mathcal{L}'}$ be the restriction of $\vdash_{\mathcal{L}}$ to $\mathcal{L}$ in the sense that $\Gamma \vdash_{\mathcal{L}'} \alpha$ iff $\Gamma \cup \{\alpha\} \subseteq Fm_{\mathcal{L}'}$. $\mathcal{L}'$ is called the **$\mathcal{L}'$-fragment** of $\mathcal{L}$.

Definition 32 ([13], Def. 2.2)**.** A **proof** in $\mathcal{L}$ is a sequence $\alpha_1, \ldots, \alpha_n$ such that for each i $(1 \leq i \leq n)$, either α_i is axiom of $\mathcal{L}$ or α_i follows from previous members of the sequence, say α_j and α_k $(j < i$, $k < i)$ as a direct consequence using rule of deduction MP. Such a proof will be referred to as a proof of α_n in $\mathcal{L}$, and α_n is said to be a **theorem** of $\mathcal{L}$.

Example 17 ([10], Exa. 1.9)**.** Let $\mathcal{H}$ be a Hilbert-style calculus on a set of formulas Fm of type $\mathbf{L}$. For every $\Gamma \subseteq Fm$ and every $\alpha \in Fm$, the relation $\Gamma \vdash_{\mathcal{HC}} \alpha$ is defined to hold if and only if there is a proof of α in $\mathcal{HC}$ from assumptions in Γ. Then $\langle \mathbf{L}, \vdash_{\mathcal{HC}} \rangle$ is a finitary logic and its theorems are the formulas that have a proof in $\mathcal{H}$ from no assumptions other than the axioms.

Remark 8. Every finitary logic can be defined by means of a Hilbert-style calculus.

Throughout this document, we are going to make use of the **Hilbert-style** presentation of a logic, in which there is only one rule of deduction, namely **modus ponens** (abbreviated MP): $\{\alpha, \alpha \to \beta\} \vdash \beta$.

2.2.1 Positive logic

Positive logic $\mathcal{LP} = \langle \mathbf{Fm}, \vdash_{\mathcal{LP}} \rangle$ is the logic over the language $\langle \wedge, \vee, \to \rangle$ of type $\langle 2, 2, 2 \rangle$ defined by the Hilbert-style calculus with the following axioms and modus ponens as the only rule:

A1 $\alpha \to (\beta \to \alpha)$

A2 $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

A3 $(\alpha \wedge \beta) \to \alpha$

A4 $(\alpha \wedge \beta) \to \beta$

A5 $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma)))$

A6 $\alpha \to (\alpha \vee \beta)$

A7 $\beta \to (\alpha \vee \beta)$

A8 $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$

Proposition 1. If $\alpha \in \mathcal{L}$ then $\vdash_{\mathcal{LP}} \alpha \to \alpha$.

Proof. [13], Example 2.7. □

Positive logic satisfies the deduction theorem.

Theorem 5. (Deduction Theorem). If $\Phi \cup \{\alpha\} \vdash_{\mathcal{LP}} \beta$, then $\Phi \vdash_{\mathcal{LP}} \alpha \to \beta$.

Proof. [13], Proposition 2.8. □

Remark 9. To prove Deduction Theorem (DT), we only need axioms **(A1)** and **(A2)** of Positive Logic and the fact that modus ponens is the only inference rule.

As an immediate consequence of the Deduction Theorem, we have:

Lemma 4. If $\alpha, \beta, \gamma \in \mathcal{L}$ then $\{\alpha \to \beta, \beta \to \gamma\} \vdash_{\mathcal{LP}} \alpha \to \gamma$.

All logics considered in this document satisfy these conditions, thus DT remains true for all logics considered below. Therefore, let's look at several extensions of $\vdash_{\mathcal{LP}}$:

1. Extending the language with $\{\neg\}$ and adding the following two axioms we will have an axiomatization of intuitionistic logic, $\vdash_{\mathcal{INT}}$:

   INT1 $\alpha \to (\neg\alpha \to \beta)$

   INT2 $(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$

2. Extending the language with $\{\neg\}$ and add the following three axioms we will have an axiomatization of classical propositional logic, $\vdash_{\mathcal{CP}}$:

   CP1 $\alpha \vee \neg\alpha$

3. Extending the language with $\{\sim\}$ and the axioms of $\vdash_{\mathcal{LP}}$ with the following four axiom schemes we obtain an axiomatization of the paraconsistent version of Nelson's logic, $\vdash_{\mathcal{N}4}$:

   A9 $(\sim\sim\alpha \to \alpha) \wedge (\alpha \to \sim\sim\alpha)$

   A10 $(\sim(\alpha \vee \beta) \to (\sim\alpha \wedge \sim\beta)) \wedge ((\sim\alpha \wedge \sim\beta) \to \sim(\alpha \vee \beta))$

   A11 $(\sim(\alpha \wedge \beta) \to (\sim\alpha \vee \sim\beta)) \wedge ((\sim\alpha \vee \sim\beta) \to \sim(\alpha \wedge \beta))$

   A12 $(\sim(\alpha \to \beta) \to (\alpha \wedge \sim\beta)) \wedge ((\alpha \wedge \sim\beta) \to \sim(\alpha \to \beta))$

4. Adding the following axiom to $\vdash_{\mathcal{N}4}$, we will obtain the logic of Nelson $\vdash_{\mathcal{N}3}$:

   A13 $\sim\alpha \to (\alpha \to \beta)$

Remark 10. We abbreviate $(\alpha \to \beta) \wedge (\beta \to \alpha) := (\alpha \leftrightarrow \beta)$. Thus,

A9' $\sim\sim\alpha \leftrightarrow \alpha$.

A10' $\sim(\alpha \vee \beta) \leftrightarrow (\sim\alpha \wedge \sim\beta)$.

A11' $\sim(\alpha \wedge \beta) \leftrightarrow (\sim\alpha \vee \sim\beta)$.

A12' $\sim(\alpha \to \beta) \leftrightarrow (\alpha \wedge \sim\beta)$.

In the paper [14], we have other logic that satisfies the Deduction Theorem, namely: quasi-Nelson logic ($\mathcal{QNL}$). This logic is obtained by adding the following axioms to Positive Logic:

QNL9 $\sim\sim(\sim\alpha \to \sim\beta) \to (\sim\alpha \to \sim\beta)$

QNL10 $(\sim\alpha \wedge \sim\beta) \leftrightarrow \sim(\alpha \vee \beta)$

QNL11 $(\sim\sim\alpha \wedge \sim\sim\beta) \leftrightarrow \sim\sim(\alpha \wedge \beta)$

QNL12 $\sim\sim\sim\alpha \to \sim\alpha$

QNL13 $\sim(\alpha \to \beta) \leftrightarrow \sim\sim(\alpha \wedge \sim\beta)$

QNL14 $\alpha \to \sim\sim\alpha$

QNL15 $(\alpha \to \beta) \to (\sim\sim\alpha \to \sim\sim\beta)$

QNL16 $\sim\alpha \to \sim(\alpha \wedge \beta)$

QNL17 $\sim(\alpha \wedge \beta) \to \sim(\beta \wedge \alpha)$

QNL18 $\sim(\alpha \wedge (\beta \wedge \gamma)) \leftrightarrow \sim((\alpha \wedge \beta) \wedge \gamma)$

QNL19 $\sim\alpha \to \sim(\alpha \wedge (\beta \vee \alpha))$

QNL20 $\sim\alpha \to \sim(\alpha \wedge (\alpha \vee \beta))$

QNL21 $\sim(\alpha \wedge (\beta \vee \gamma)) \leftrightarrow \sim((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$

QNL22 $\sim(\alpha \vee (\beta \wedge \gamma)) \leftrightarrow \sim((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$

QNL23 $\sim\alpha \leftrightarrow \sim(\alpha \wedge (\beta \to \beta))$

QNL24 $\sim(\alpha \to \alpha) \to \beta$

QNL25 $(\sim\alpha \to \sim\beta) \to (\sim(\alpha \wedge \beta) \to \sim\beta)$

QNL26 $(\sim\alpha \to \sim\beta) \to ((\sim\gamma \to \sim\theta) \to (\sim(\alpha \wedge \gamma) \to \sim(\beta \wedge \theta)))$

2.2.2 Full Lambek calculus with exchange and weakening

Since $\mathcal{N}3$ and $\mathcal{QNL}$ are obtained as axiomatic extensions of $\mathcal{FL}_{ew}$, it is worth presenting a calculation for this logic. Thus, the logic $\mathcal{FL}_{ew} = \langle \mathbf{Fm}, \vdash_{\mathcal{FL}_{ew}} \rangle$ is the logic over the language $\langle \vee, \wedge, \Rightarrow, *, \bot, \top \rangle$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ defined by the Hilbert-style calculus with the following axioms and modus ponens as the only rule:

(A1) $(\alpha \Rightarrow \beta) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \gamma))$

(A2) $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow (\beta \Rightarrow (\alpha \Rightarrow \gamma))$

(A3) $\alpha \Rightarrow (\beta \Rightarrow \alpha)$

(A4) $\alpha \Rightarrow (\beta \Rightarrow (\alpha * \beta))$

(A5) $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha * \beta) \Rightarrow \gamma)$

(A6) $(\alpha \wedge \beta) \Rightarrow \alpha$

(A7) $(\alpha \wedge \beta) \Rightarrow \beta$

(A8) $(\alpha \Rightarrow \beta) \Rightarrow ((\alpha \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow (\beta \wedge \gamma)))$

(A9) $\alpha \Rightarrow (\alpha \vee \beta)$

(A10) $\beta \Rightarrow (\alpha \vee \beta)$

(A11) $(\alpha \Rightarrow \gamma) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow ((\alpha \vee \beta) \Rightarrow \gamma))$

(A12) $\top$

(A13) $\bot \Rightarrow \alpha$

2.3 Algebraizable Logics

In this section we formally define the concept of *algebraic semantics* and *algebraizable logics* that are extensively used in this document.

Given a logic $\mathcal{L}$, we are interested in associating its relation $\vdash_{\mathcal{L}}$ to a relation $\vDash_{\mathcal{K}}$ between sets of equations and equations in the language of a class of algebras $\mathcal{K}$, in a way that we can study $\vDash_{\mathcal{K}}$ to answer questions about $\vdash_{\mathcal{L}}$ and vice-versa. This relation $\vDash_{\mathcal{K}}$ is used, then, to define what an algebraic semantics for a logic is.

Definition 33 ([10], Def. 1.69)**.** The **relative equational consequence** associated with a class $\mathcal{K}$ of algebras is the relation $\vDash_{\mathcal{K}} \subseteq \wp(Eq) \times Eq$ defined next: given $\Theta \cup \{\alpha = \beta\} \subseteq Eq$,

$$\Theta \vDash_{\mathcal{K}} \alpha = \beta \quad \overset{def}{\Longleftrightarrow} \quad \begin{array}{r} \text{For every} \quad \mathbf{A} \in \mathcal{K} \quad \text{and every} \quad h \in \mathtt{Hom}(\mathbf{Fm}, \mathbf{A}), \\ \text{if} \quad h(\phi) = h(\psi) \quad \text{for all} \quad \phi = \psi \in \Theta, \quad \text{then} \quad h(\alpha) = h(\beta). \end{array}$$

The relation between the logic and relative equational consequence of a class of algebras, is effected by means of two transformers, a generalization of the functions $\alpha \longmapsto \alpha = \top$ and $\alpha = \beta \longmapsto \{\alpha \to \beta, \beta \to \alpha\}$ that transform a formula into an equation and an equation into a set of formulas.

Definition 34 ([10], Def. 3.1)**.** A **transformer** from formulas to sets of equations is any function $\tau : Fm \to \wp(Eq)$. It is extended to a function $\tau : \wp(Fm) \to \wp(Eq)$ by setting, for any $\Gamma \subseteq Fm$, $\tau\Gamma := \bigcup_{\gamma \in \Gamma} \tau\gamma$.

Definition 35 ([10], Def. 3.2)**.** A transformer τ is **structural** when it commutes with substitutions in the sense that $\tau\sigma = \sigma\tau$ for every substitution σ.

Proposition 2 ([10], Prop. 3.3)**.** A transformer from formulas to equations τ is structural if and only if there is a set of equations $E(x) \subseteq Eq$ in at most one variable x such that $\tau\alpha = E\alpha$ for all $\alpha \in Fm$.

Definition 36 ([10], Def. 3.4)**.** Let $\mathcal{L}$ be a logic, $\mathcal{K}$ a class of algebras, and τ a structural transformer. The class $\mathcal{K}$ is an **algebraic semantics** for $\mathcal{L}$ when the following condition is satisfied, for all $\Gamma \cup \{\alpha\} \subseteq F_m$:

(**ALG1**) $\Gamma \vdash_{\mathcal{L}} \alpha \Longleftrightarrow \tau\Gamma \vDash_{\mathcal{K}} \tau\alpha$

The set $E(x)$ corresponding to the transformer τ is called the set of **defining equations**.

In the same way we defined a transformer from sets of formulas to sets of equations, we define a transformer from set of equations to sets of formulas.

Definition 37 ([10]). A **transformer** from equations to sets of formulas is any function $\rho : Eq \to \wp(Fm)$. It is extended to a function $\rho : \wp(Eq) \to \wp(Fm)$ by setting, for any $\Theta \subseteq Eq$, $\rho\Theta := \bigcup_{\delta=\epsilon\in\Theta} \Delta(\delta, \epsilon)$.

Definition 38 ([10]). A transformer ρ is **structural** when it commutes with unions.

Proposition 3 ([10]). A transformer from equations to formulas ρ is structural if and only if there is a set of formulas $\Delta(x, y)$ in at most two variables x, y such that $\rho(\alpha = \beta) = \Delta(\alpha, \beta)$ for all $\alpha = \beta \in Eq$.

The set $\Delta(x, y)$ corresponding to the transformer ρ is called set of **equivalence formulas**.

Definition 39 ([10], Def. 3.11). A logic $\mathcal{L}$ is **algebraizable** when there is a class $\mathcal{K}$ of algebras and structural transformers τ, ρ (from sets of formulas to sets of equations and from sets of equations to sets of formulas, respectively) such that the following conditions are satisfied, for all $\Gamma \cup \{\alpha\} \subseteq Fm$ and all $\Theta \cup \{\delta = \epsilon\} \subseteq Eq$:

(ALG1) $\Gamma \vdash_{\mathcal{L}} \alpha \iff \tau\Gamma \vDash_{\mathcal{K}} \tau\alpha$

(ALG2) $\Theta \vDash_{\mathcal{K}} \delta = \epsilon \iff \rho\Theta \vdash_{\mathcal{L}} \rho(\delta = \epsilon)$

(ALG3) $\alpha \dashv\vdash_{\mathcal{L}} \rho\tau\alpha$

(ALG4) $\delta = \epsilon \vDash_{\mathcal{K}} \tau\rho(\epsilon = \delta)$ and $\tau\rho(\delta = \epsilon) \vDash_{\mathcal{K}} \delta = \epsilon$

The transformers τ and ρ are said to **witness** the algebraizability of $\mathcal{L}$ with respect to the class $\mathcal{K}$.

Proposition 4 ([10], Prop. 3.12). A logic $\mathcal{L}$ is algebraizable if and only if there is a class $\mathcal{K}$ of algebras and there are structural transformers τ, ρ such that conditions **(ALG1)** and **(ALG4)** are satisfied; or, equivalently, conditions **(ALG2)** and **(ALG3)**.

In the next theorem we will show that although a logic $\mathcal{L}$ can be algebraizable with different sets of defining equations, equivalence formulas and classes of algebras, its algebraizations are in a certain sense the same, and we will use this fact to choose one among all classes of algebras $\mathcal{K}$ such that $\mathcal{L}$ is algebraizable with respect to it.

Theorem 6 ([10], Thm. 3.17)**.** Let $\mathcal{L}$ is algebraizable logic with respect to a class $\mathcal{K}$, with defining equations $E(x)$ and equivalence formulas $\Delta(x, y)$. The logic $\mathcal{L}'$ is algebraizable with respect to a class $\mathcal{K}'$, with defining equations $E'(x)$ and equivalence formulas $\Delta'(x, y)$ if and only if the following conditions are satisfied:

1. $\vDash_{\mathcal{K}} = \vDash_{\mathcal{K}'}$.
2. $\Delta(x, y) \dashv\vdash_{\mathcal{L}} \Delta'(x, y)$.
3. $E(x) \vDash_{\mathcal{K}} E'(x)$ and $E'(x) \vDash_{\mathcal{K}} E(x)$.

Definition 40 ([10], Def. 3.21)**.** Let $\mathcal{L}$ be an algebraizable logic. Its **equivalent algebraic semantics** is the largest class of algebras $\mathcal{K}$ such that $\mathcal{L}$ is algebraizable with respect to $\mathcal{K}$.

Given an algebraizable logic $\mathcal{L}$, we use the notation $\mathtt{Alg}^*(\mathcal{L})$ to denote its equivalent algebraic semantics.

Definition 39 allows us to establish algebraizability of a logic only with prior knowledge of the class $\mathcal{K}$ and of the transformers. There is also syntactic criterion that allows checking whether a given pair of transformers witnesses the algebraizability of a given logic by just looking at their behaviour regarding the consequence relation of the logic. Such a criterion is sometimes qualified as an intrinsic characterization of algebraizability, but in fact it is only partially so, as it still depends on knowledge of the transformers. This is content of the next theorem.

Theorem 7 ([10], Thm. 3.19)**.** A logic $\mathcal{L}$ is **algebraizable** if and only if there are equations $E(x) \subseteq Eq$ and formulas $\Delta(x, y) \subseteq Fm$, such that $\mathcal{L}$ satisfies the following five conditions:

(R) $\vdash_{\mathcal{L}} \Delta(x, x)$

(Sym) $\Delta(x, y) \vdash_{\mathcal{L}} \Delta(y, x)$

(Trans) $\Delta(x, y) \cup \Delta(y, z) \vdash_{\mathcal{L}} \Delta(x, z)$

(Re) $\bigcup_{i=1}^{n} \Delta(x_i, y_i) \vdash_{\mathcal{L}} \Delta(\lambda\, x_1 \dots x_n, \lambda\, y_1 \dots y_n)$ for all $\lambda \in \mathbf{L}$, with $n = \mathrm{ar}\, \lambda$

(ALG3) $x \dashv\vdash_{\mathcal{L}} \Delta(E(x))$

The five conditions above can be replaced by the conditions below:

(Ref) $\vdash_{\mathcal{L}} \Delta(x, x)$

(MP) $x, \Delta(x, y) \vdash_{\mathcal{L}} y$

(Alg) $x \dashv\vdash_{\mathcal{L}} \Delta(E(x))$

(Cong) for each n-ary connective λ, $\bigcup_{i=1}^{n} \Delta(x_i, y_i) \vdash_{\mathcal{L}} \Delta(\lambda(x_1, \dots, x_n), \lambda(y_1, \dots, y_n))$.

The algebraizability of a large number of logics has been shown in the literature by using Theorem 7, either directly or through the next straightforward application, which settles the issue of the algebraizability of extensions, fragments and expansions:

Proposition 5 ([10], Prop. 3.31)**.** Let $\mathcal{L}$ be an algebraizable logic with respect to $\mathcal{K}$ with transformers τ, ρ.

1. Every axiomatic extension $\mathcal{L}'$ of $\mathcal{L}$ is is algebraizable as well, with respect to a subclass $\mathcal{K}'$ of $\mathcal{K}$ and with the same transformers.
2. If $\mathbf{L}'$ is a fragment of the language of $\mathcal{L}$ such that $\tau x \subseteq Eq_{\mathbf{L}'}$ and $\rho(x = y) \subseteq Fm_{\mathbf{L}'}$, then $\mathcal{L}' := \mathcal{L} \upharpoonright \mathbf{L}'$, the $\mathbf{L}'$-fragment of $\mathcal{L}$, is algebraizable with respect to the class $\mathcal{K} \upharpoonright \mathbf{L}'$ and with the same transformers.
3. If L' is an expansion of $\mathcal{L}$ such that $\vdash_{\mathcal{L}'}$ satisfies condition **(Re)** for the additional connectives, then $\mathcal{L}'$ is algebraizable, with the same transformers.

Finally, there is a simple algorithm for converting any axiomatization of $\mathcal{L}$ into a basis for the quasi-equations of its unique equivalent algebraic semantics.

Theorem 8 ([2], Thm. 2.17)**.** Let $\mathcal{L}$ be a logic given be a set of axioms **Ax** and a set of inference rules **Ru**. Assume $\mathcal{L}$ is algebraizable with equivalence formulas Δ and defining equations $\delta = \epsilon$. Then the unique equivalent quasi-variety semantics for $\mathcal{L}$ is axiomatized by the following equations

(i) $\delta(\alpha) = \epsilon(\alpha)$ for each $\alpha \in \mathbf{Ax}$.

(ii) $\delta(p\Delta p) = \epsilon(p\Delta p)$.

together with the following quasi-equations

(iii) $\delta(\beta_0) = \epsilon(\beta_0) \wedge \ldots \wedge \delta(\beta_{n-1}) = \epsilon(\beta_{n-1}) \rightarrow \delta(\alpha) = \epsilon(\alpha)$, for each $\langle\{\beta_0, \ldots, \beta_{n-1}, \alpha\rangle \in \mathbf{Ru}$.

(iv) $\delta(p\Delta q) = \epsilon(p\Delta q) \rightarrow p = q$.

For more details about this theorem and the notations involved, we suggest the book [2].

3 Quasi-Nelson algebras and Nuclei

In this chapter we shall consider algebras that result from adding a modal-like operator subreducts of Heyting algebras; such operators are known as *nuclei* (or modal operators). We will consider the following two different, but essentially equivalent definitions, for a *nucleus*, which depend on what other operations are available in the algebra.

Definition 41 ([23], Def. 2.5)**.** Let $\mathbf{A}$ be an algebra having a reduct $\langle A; \wedge, 0\rangle$ that is a (meet-) semilattice with order $\leq$ and minimum 0. We shall say that an operation $\Box : A \to A$ is a **nucleus** on $\mathbf{A}$ if the following equations are satisfied:

(i) $x \leq \Box x = \Box\Box x$.

(ii) $\Box(x \wedge y) = \Box x \wedge \Box y$.

(iii) $\Box 0 = 0$.

Remark 11. The equations of Definition 41 entail that, if the order $\leq$ has a maximum element 1, then $\Box 1 = 1$; so, $\Box$ is indeed a modal-like operator in that it preserves all finite meets.

Definition 42 ([23], Def. 2.6)**.** Given an algebra having a bounded Hilbert algebra reduct $\langle H; \to, 0, 1\rangle$, we say that an operation $\Box : H \to H$ is a **nucleus** on $\mathbf{H}$ if:

(i) $x \leq \Box x = \Box\Box x$.

(ii) $\Box(x \to y) = \Box x \to \Box y$.

(iii) $\Box 0 = 0$.

Definition 43 ([25], Def. 4.1)**.** An algebra $\mathbf{A} = \langle A;\ \wedge, \vee, \to, \sim, 0, 1\rangle$ of type $\langle 2, 2, 2, 1, 0, 0\rangle$ is called a **quasi-Nelson algebra** if the following hold.

(QN1) The reduct $\langle A;\ \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice with order $\leq$.

(QN2) The relation $\preceq$ on A defined for all $a,\ b \in A$ by $a \preceq b$ iff $a \to b = 1$ is a quasiorder on A.

(QN3) The relation $\equiv\ := \preceq \cap (\preceq)^{-1}$ is a congruence on the reduct $\langle A;\ \wedge, \vee, \to, 0, 1\rangle$ and the quotient algebra $\mathbf{A}_+ = \langle A;\ \wedge, \vee, \to, 0, 1\rangle/\equiv$ is a Heyting algebra.

(QN4) For all $a,\ b \in A$, it holds that $\sim(a \to b) \equiv \sim\sim(a \wedge \sim b)$.

(QN5) For all $a,\ b \in A$, it holds that $a \leq b$ iff $a \preceq b$ and $\sim b \preceq \sim a$.

(QN6) For all $a,\ b \in A$,

(QN6.1) $\sim(\sim a \to \sim b) \equiv \sim a \to \sim b$.

(QN6.2) $\sim(a \vee b) \equiv \sim a \wedge \sim b$.

(QN6.3) $\sim\sim a \wedge \sim\sim b \equiv \sim(a \wedge b)$.

(QN6.4) $\sim a \equiv \sim\sim\sim a$.

(QN6.5) $a \preceq \sim\sim a$.

(QN6.6) $a \wedge \sim a \preceq 0$.

An alternative language in which quasi-Nelson algebras have been considered is $\{\wedge, \vee, \to, 0, 1\}$, in which the residuated implication $\Rightarrow$ (in this context known as the *strong implication*) is replaced by the *weak implication* $\to$, defining: $x \Rightarrow y := (x \to y) \wedge (\sim y \to \sim x)$. In turn, the weak implication is definable via the strong one by the term $x \to y := x \Rightarrow (x \Rightarrow y)$. Based on these equivalences, and depending on convenience, we can therefore employ the strong or weak implication to express the properties of quasi-Nelson algebras we are interested in.

A fundamental result on quasi-Nelson algebras (and some of their subreducts) is the twist representation, which we now proceed to introduce.

Definition 44 ([23], Def. 2.9)**.** Let $\mathbf{H} = \langle H;\ \wedge, \vee, \rightarrow, \Box, 0, 1\rangle$ be a Heyting algebra of type $\langle 2, 2, 2, 1, 0, 0\rangle$ with a nucleus. Define the algebra $\mathbf{H}^{\bowtie} = \langle H^{\bowtie};\ \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ with universe:

$$H^{\bowtie} := \{\langle a_1, a_2\rangle \in H \times H;\ \ a_2 = \Box a_2, a_1 \wedge a_2 = 0\}$$

and operations given, for all $\langle a_1, a_2\rangle$, $\langle b_1, b_2\rangle \in H \times H$, by:

$$\begin{aligned}
0 &:= \langle 0, 1\rangle \\
1 &:= \langle 1, 0\rangle \\
\langle a_1, a_2\rangle * \langle b_1, b_2\rangle &:= \langle a_1 \wedge b_1, (a_1 \rightarrow b_2) \wedge (b_1 \rightarrow a_2)\rangle \\
\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle &:= \langle a_1 \wedge b_1, \Box(a_2 \vee b_2)\rangle \\
\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle &:= \langle a_1 \vee b_1, a_2 \wedge b_2\rangle \\
\langle a_1, a_2\rangle \Rightarrow \langle b_1, b_2\rangle &:= \langle (a_1 \rightarrow b_1) \wedge (b_2 \rightarrow a_2), \Box a_1 \wedge b_2\rangle
\end{aligned}$$

A **quasi-Nelson twist-algebra** over $\mathbf{H}$ is any subalgebra $\mathbf{A} \leq \mathbf{H}^{\bowtie}$ satisfying $\pi_1[A] = H$.

Remark 12 ([23])**.** Every quasi-Nelson twist-algebra is a quasi-Nelson algebra on which the negation is given by $\sim x := x \Rightarrow 0$ and the weak implication by $x \rightarrow y := x \Rightarrow (x \Rightarrow y)$.

Based on the above remark, we have

$$\begin{cases} \sim\langle a_1, a_2\rangle = \langle a_2, \Box a_1\rangle \\ \langle a_1, a_2\rangle \rightarrow \langle b_1, b_2\rangle = \langle a_1 \rightarrow b_1, \Box a_1 \wedge b_2\rangle \end{cases}$$

Given a quasi-Nelson algebra $\mathbf{A} = \langle A;\ \wedge, \vee, *, \Rightarrow, 0, 1\rangle$, we can define the relation $\equiv$ as:

$$a \equiv b \quad \text{iff} \quad a \rightarrow b = b \rightarrow a = 1;\ \ \forall\, a, b \in A$$

This relation $\equiv$ is compatible with the operations $\langle \wedge, \vee, *, \rightarrow\rangle$, though not necessarily with $\Rightarrow$ and $\sim$, giving us a quotient $\langle A/{\equiv};\ \wedge, \vee, *, \rightarrow, 0, 1\rangle$. Since $a \equiv b$

entails $\sim\sim a \equiv \sim\sim b$ for all a, $b \in A$, one can enrich the quotient aforementioned with a well-defined operation, given by $\Box[a] := [\sim\sim a]$ for each class $[a] \in A/\equiv$, which turns out to be a nucleus. Letting $A_{\bowtie} := \langle A/\equiv;\ \wedge, \vee, \to, \Box, 0, 1\rangle$, we have a Heyting algebra with a nucleus and we can construct the twist-algebra $(\mathbf{A}_{\bowtie})^{\bowtie}$ as prescribed by Definition 44, obtaining the followings results.

Theorem 9 ([23], Thm. 2.10)**.** Every quasi-Nelson algebra $\mathbf{A}$ embeds into the quasi-Nelson twist-algebra $(\mathbf{A}_{\bowtie})^{\bowtie}$ with the map ι given by $\iota(a) = \langle [a], [\sim a]\rangle$ for all $a \in A$.

Proposition 6 ([23], Prop. 2.11)**.** Every quasi-Nelson algebra satisfies the following identity: $x \Rightarrow y = (x \to y) * ((x \to y) \to (\sim y \to \sim x))$.

The proposition above is especially significant in the present context because it entails that the $\{*, \Rightarrow, \sim\}$-fragment of quasi-Nelson logic is term equivalent to the $\{*, \to, \sim\}$-fragment. This fact will be used in the chapter on the fragments of quasi-Nelson logic.

4 QN4-lattices and their logic

The class of quasi-N4-lattices (QN4-lattices) was introduced as a common generalization of quasi-Nelson algebras (QNA) and N4-lattices, in such a way that N4-lattices are precisely the QN4-lattices satisfying the double negation law ($\sim\sim x = x$) and QNA are the QN4-lattices satisfying $(x \wedge \sim x) \to y = ((x \wedge \sim x) \to y) \to ((x \wedge \sim x) \to y)$, the explosive law. For more details about QN4-lattices, see [22].

In this chapter we introduce, via a Hilbert-style presentation, a logic ($\mathcal{L}_{\mathbf{QN4}}$) whose algebraic semantics is a class of algebras that we show to be term-equivalent to QN4-lattices. The result is obtained by showing that the calculus introduced by us is algebraizable in the sense of Blok and Pigozzi, and its equivalent algebraic semantics is term-equivalent to the class of QN4-lattices.

4.1 QN4-lattices

In this section we recall two equivalent presentations of quasi-N4-lattices; these will be used to establish the equivalence between the two alternative algebraic semantics for the logic $\mathcal{L}_{\mathbf{QN4}}$, which is introduced in the next section.

We shall refer to an algebra $\mathbf{B} = \langle B;\ \wedge, \vee, \to, \Box\rangle$ as to a nuclear Brouwerian algebra, where $\Box$ is a nucleus in the sense of Definition 41.

Definition 45 ([22], Def. 2.2)**.** Let $\mathbf{B} = \langle B;\ \wedge, \vee, \to, \Box\rangle$ be a nuclear Brouwerian algebra. The algebra $\mathbf{B}^{\bowtie} = \langle B \times B;\ \wedge, \vee, \to, \sim\rangle$ is defined as follows. For all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in$

$B \times B$, we have:

$$\sim\langle a_1, a_2\rangle = \langle a_2, \Box a_1\rangle$$

$$\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle = \langle a_1 \wedge b_1, \Box(a_2 \vee b_2)\rangle$$

$$\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle = \langle a_1 \vee b_1, a_2 \wedge b_2\rangle$$

$$\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle = \langle a_1 \to b_1, \Box a_1 \wedge b_2\rangle$$

A **quasi-N4 twist-structure A** over **B** is a subalgebra of $\mathbf{B}^{\bowtie}$ satisfying the following properties: $\pi_1[A] = B$ and $\Box a_2 = a_2$ for all $\langle a_1, a_2\rangle \in A$.

Given an algebra **A** having an operation $\to$ and elements $a, b \in A$, we shall abbreviate $|a| := a \to a$, and define the relations $\equiv$ and $\preceq$ as follows. We let $a \preceq b$ iff $a \to b = |a \to b|$, and $\equiv := (\preceq \cap (\preceq)^{-1})$. Thus one has $a \equiv b$ iff ($a \preceq b$ and $b \preceq a$).

Definition 46 ([22], Def. 3.2)**.** A **quasi-N4-lattice** (QN4-lattice) is an algebra $\mathbf{A} = \langle A;\ \wedge, \vee, \to, \sim\rangle$ of type $\langle 2, 2, 2, 1\rangle$ satisfying the following properties:

(QN4a) The reduct $\langle A;\ \wedge, \vee\rangle$ is a distributive lattice with lattice order $\leq$.

(QN4b) The relation $\equiv := (\preceq \cap (\preceq)^{-1})$ is a congruence on the reduct $\langle A;\ \wedge, \vee, \to\rangle$ and the quotient $B(\mathbf{A}) = \langle A;\ \wedge, \vee, \to\rangle/{\equiv}$ is a Brouwerian algebra. The operator $\Box$ given by $\Box[a] := (\sim\sim a/{\equiv})$ for all $a \in A$ is a nucleus, so the algebra $\langle B(\mathbf{A}), \Box\rangle$ is a nuclear Brouwerian algebra.

(QN4c) For all $a, b \in A$, it holds that $a \leq b$ iff $a \preceq b$ and $\sim b \preceq \sim a$.

(QN4d) For all $a, b \in A$, it holds that $\sim(a \to b) \equiv \sim\sim(a \wedge \sim b)$.

(QN4e) For all $a, b \in A$,

- **(QN4e.1)** $a \leq \sim\sim a$.
- **(QN4e.2)** $\sim a = \sim\sim\sim a$.
- **(QN4e.3)** $\sim(a \vee b) = \sim a \wedge \sim b$.
- **(QN4e.4)** $\sim\sim a \wedge \sim\sim b = \sim\sim(a \wedge b)$.

The preceding definition is a straightforward generalization of Odintsov's [19] definition of N4-lattices; indeed, as observed in [22, Proposition 3.8], a quasi-N4-lattice $\mathbf{A}$ is an N4-lattice if and only if $\mathbf{A}$ it is involutive, that is, $\sim\sim a \leq a$ for all $a \in A$. Similarly, a quasi-Nelson algebra may be defined as a quasi-N4-lattice $\mathbf{A}$ that satisfies the explosive equality, $a \wedge \sim a \preceq b$, for all $a, b \in A$.

Theorem 10 ([22], Thm. 3.3)**.** Every quasi-N4-lattice $\mathbf{A}$ is isomorphic to a twist-structure over $\langle B(\mathbf{A}), \Box \rangle$ by the map $\iota : A \to (A/\equiv) \times (A/\equiv)$ given by $\iota(a) := \langle a/\equiv, \sim a/\equiv \rangle$ for all $a \in A$.

In the proposition below we see that the non-equational presentation for QN4-lattices given in Definition 46 can be replaced with an equational one, entailing that QN4-lattices form a variety of algebras.

Proposition 7 ([22], Prop. 3.7)**.** Items **(QN4b)** and **(QN4c)** in Definition 46 can be equivalently replaced by the following equations:

1. $|x| \to y = y$.
2. $(x \wedge y) \to x = |(x \wedge y) \to x|$.
3. $(x \wedge y) \to z = x \to (y \to z)$.
4. $(x \Leftrightarrow y) \to x = (x \Leftrightarrow y) \to y$.
5. $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.
6. $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.
7. $(x \to y) \wedge (y \to z) \preceq x \to z$.
8. $x \to y \preceq x \to (y \vee z)$.
9. $x \to (y \to z) = (x \to y) \to (x \to z)$.
10. $x \to y \preceq \sim\sim x \to \sim\sim y$.

4.2 A Hilbert-style calculus

In this section we introduce a Hilbert-style calculus that determines a logic, henceforth denoted by $\mathcal{L}_{\mathbf{QN4}}$. Our aim is to show that $\mathcal{L}_{\mathbf{QN4}}$ is algebraizable, and that its equivalent algebraic semantics is term-equivalent to the class of QN4-lattices.

The Hilbert-system for $\mathcal{L}_{\mathbf{QN4}}$ consists of the following axiom schemes together with the single inference rule of *modus ponens* (MP): $\alpha, \alpha \to \beta \vdash \beta$.

Ax1 $\alpha \to (\beta \to \alpha)$

Ax2 $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

Ax3 $(\alpha \wedge \beta) \to \alpha$

Ax4 $(\alpha \wedge \beta) \to \beta$

Ax5 $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma)))$

Ax6 $\alpha \to (\alpha \vee \beta)$

Ax7 $\beta \to (\alpha \vee \beta)$

Ax8 $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$

Ax9 ${\sim}(\alpha \vee \beta) \leftrightarrow ({\sim}\alpha \wedge {\sim}\beta)$

Ax10 ${\sim}(\alpha \to \beta) \leftrightarrow {\sim}{\sim}(\alpha \wedge {\sim}\beta)$

Ax11 ${\sim}(\alpha \wedge (\beta \wedge \gamma)) \leftrightarrow {\sim}((\alpha \wedge \beta) \wedge \gamma)$

Ax12 ${\sim}(\alpha \wedge (\beta \vee \gamma)) \leftrightarrow {\sim}((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$

Ax13 ${\sim}(\alpha \vee (\beta \wedge \gamma)) \leftrightarrow {\sim}((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$

Ax14 ${\sim}{\sim}(\alpha \wedge \beta) \leftrightarrow ({\sim}{\sim}\alpha \wedge {\sim}{\sim}\beta)$

Ax15 $\alpha \to {\sim}{\sim}\alpha$

Ax16 $\alpha \to ({\sim}\alpha \to {\sim}(\alpha \to \alpha))$

Ax17 $(\alpha \to \beta) \to ({\sim}{\sim}\alpha \to {\sim}{\sim}\beta)$

Ax18 ${\sim}\alpha \to {\sim}(\alpha \wedge \beta)$

Ax19 $\sim(\alpha \wedge \beta) \to \sim(\beta \wedge \alpha)$

Ax20 $(\sim\alpha \to \sim\beta) \to (\sim(\alpha \wedge \beta) \to \sim\beta)$

Ax21 $(\sim\alpha \to \sim\beta) \to ((\sim\gamma \to \sim\theta) \to (\sim(\alpha \wedge \gamma) \to \sim(\beta \wedge \theta)))$

Ax22 $\sim\sim\sim\alpha \to \sim\alpha$

It should be noted that the Deduction Theorem holds for $\mathcal{L}_{\mathbf{QN4}}$.

4.3 $\mathcal{L}_{\mathbf{QN4}}$ is BP-Algebraizable

In this section we prove that the calculus introduced in the previous section is algebraizable in sense of Blok and Pigozzi. Using this result, we will axiomatize the equivalent algebraic semantics of $\mathcal{L}_{\mathbf{QN4}}$ via the algorithm in Theorem 8 and show that it is term-equivalent to the class of QN4-lattices.

Theorem 11. $\mathcal{L}_{\mathbf{QN4}}$ is BP-algebraizable with $E(\alpha) := \{\alpha = \alpha \to \alpha\}$ and $\Delta(\alpha, \beta) := \{\alpha \to \beta, \beta \to \alpha, \sim\alpha \to \sim\beta, \sim\beta \to \sim\alpha\}$.

Proof. By Theorem 7, to prove (**Ref**), it is necessary to show that $\vdash_{\mathcal{L}_{\mathbf{QN4}}} \{\alpha \to \alpha, \sim\alpha \to \sim\alpha\}$, and it is Proposition 1. (**MP**): $\alpha, \{\alpha \to \beta, \beta \to \alpha, \sim\alpha \to \sim\beta, \sim\beta \to \sim\alpha\} \vdash_{\mathcal{L}_{\mathbf{QN4}}} \beta$ is a straightforward consequence of modus ponens. As to (**Alg**), it suffices to prove that $\alpha \dashv\vdash_{\mathcal{L}_{\mathbf{QN4}}} \{\alpha \to (\alpha \to \alpha), (\alpha \to \alpha) \to \alpha, \sim\alpha \to \sim(\alpha \to \alpha), \sim(\alpha \to \alpha) \to \sim\alpha\}$. From right to left, thanks to Proposition 1 and using MP, we infer the desired result. From left to right, we will prove that: (i) $\alpha \vdash_{\mathcal{L}_{\mathbf{QN4}}} \alpha \to (\alpha \to \alpha)$, we have it by instantiating **Ax1**; (ii) $\alpha \vdash_{\mathcal{L}_{\mathbf{QN4}}} (\alpha \to \alpha) \to \alpha$, follows from **Ax1** and MP; (iii) $\alpha \vdash_{\mathcal{L}_{\mathbf{QN4}}} \sim\alpha \to \sim(\alpha \to \alpha)$ is logical consequence of **Ax16** and modus ponens; (iv) $\alpha \vdash_{\mathcal{L}_{\mathbf{QN4}}} \sim(\alpha \to \alpha) \to \sim\alpha$, we have:

1. α	Premise
2. $\sim(\alpha \to \alpha) \to \sim\sim(\alpha \wedge \sim\alpha)$	Ax10 $(\to)$
3. $\sim\sim(\alpha \wedge \sim\alpha) \to (\sim\sim\alpha \wedge \sim\sim\sim\alpha)$	Ax14 $(\to)$
4. $\sim(\alpha \to \alpha) \to (\sim\sim\alpha \wedge \sim\sim\sim\alpha)$	Lemma 4, 2, 3

5. $(\sim\sim\alpha \wedge \sim\sim\sim\alpha) \rightarrow \sim\sim\sim\alpha$ Ax4
6. $\sim(\alpha \rightarrow \alpha) \rightarrow \sim\sim\sim\alpha$ Lemma 4, 4, 5
7. $\sim\sim\sim\alpha \rightarrow \sim\alpha$ Ax22
8. $\sim(\alpha \rightarrow \alpha) \rightarrow \sim\alpha$ Lemma 4, 6, 7

As to (**Cong**), we need to prove for each connective $\lambda \in \{\sim, \wedge, \vee, \rightarrow\}$.

For ($\sim$), we need to prove that:

$$\{\alpha \rightarrow \beta, \beta \rightarrow \alpha, \sim\alpha \rightarrow \sim\beta, \sim\beta \rightarrow \sim\alpha\} \vdash_{\mathcal{L}_{\mathbf{QN4}}} \sim\alpha \rightarrow \sim\beta \tag{4.1}$$

$$\{\alpha \rightarrow \beta, \beta \rightarrow \alpha, \sim\alpha \rightarrow \sim\beta, \sim\beta \rightarrow \sim\alpha\} \vdash_{\mathcal{L}_{\mathbf{QN4}}} \sim\beta \rightarrow \sim\alpha \tag{4.2}$$

$$\{\alpha \rightarrow \beta, \beta \rightarrow \alpha, \sim\alpha \rightarrow \sim\beta, \sim\beta \rightarrow \sim\alpha\} \vdash_{\mathcal{L}_{\mathbf{QN4}}} \sim\sim\alpha \rightarrow \sim\sim\beta \tag{4.3}$$

$$\{\alpha \rightarrow \beta, \beta \rightarrow \alpha, \sim\alpha \rightarrow \sim\beta, \sim\beta \rightarrow \sim\alpha\} \vdash_{\mathcal{L}_{\mathbf{QN4}}} \sim\sim\beta \rightarrow \sim\sim\alpha \tag{4.4}$$

In (4.1) and (4.2), the conclusion follows directly from the premises. Also, in (4.3) and (4.4), the conclusion can be inferred from **Ax17** and MP.

Now consider the following sets, $\Gamma_1 = \{\alpha_1 \rightarrow \beta_1, \beta_1 \rightarrow \alpha_1, \sim\alpha_1 \rightarrow \sim\beta_1, \sim\beta_1 \rightarrow \sim\alpha_1\}$ and $\Gamma_2 = \{\alpha_2 \rightarrow \beta_2, \beta_2 \rightarrow \alpha_2, \sim\alpha_2 \rightarrow \sim\beta_2, \sim\beta_2 \rightarrow \sim\alpha_2\}$.

For ($\wedge$), we need to prove that:

$$\Gamma_1 \cup \Gamma_2 \vdash (\alpha_1 \wedge \alpha_2) \rightarrow (\beta_1 \wedge \beta_2) \tag{4.5}$$

$$\Gamma_1 \cup \Gamma_2 \vdash (\beta_1 \wedge \beta_2) \rightarrow (\alpha_1 \wedge \alpha_2) \tag{4.6}$$

$$\Gamma_1 \cup \Gamma_2 \vdash \sim(\alpha_1 \wedge \alpha_2) \rightarrow \sim(\beta_1 \wedge \beta_2) \tag{4.7}$$

$$\Gamma_1 \cup \Gamma_2 \vdash \sim(\beta_1 \wedge \beta_2) \rightarrow \sim(\alpha_1 \wedge \alpha_2) \tag{4.8}$$

The item (4.6), follows the same line of reasoning from (4.5), so we will only show item (4.5).

1. $\alpha_1 \rightarrow \beta_1$ Premise
2. $\alpha_2 \rightarrow \beta_2$ Premise
3. $(\alpha_1 \wedge \alpha_2) \rightarrow \alpha_1$ Ax3

4. $(\alpha_1 \wedge \alpha_2) \to \beta_1$ — Lemma 4, 1, 3
5. $(\alpha_1 \wedge \alpha_2) \to \alpha_2$ — Ax4
6. $(\alpha_1 \wedge \alpha_2) \to \beta_2$ — Lemma 4, 2, 5
7. $((\alpha_1 \wedge \alpha_2) \to \beta_1) \to (((\alpha_1 \wedge \alpha_2) \to \beta_2) \to ((\alpha_1 \wedge \alpha_2) \to (\beta_1 \wedge \beta_2)))$ — Ax5
8. $((\alpha_1 \wedge \alpha_2) \to \beta_2) \to ((\alpha_1 \wedge \alpha_2) \to (\beta_1 \wedge \beta_2))$ — MP, 4, 7
9. $(\alpha_1 \wedge \alpha_2) \to (\beta_1 \wedge \beta_2)$ — MP, 6, 8

The derivation of (4.7) and (4.8) are straightforward and make use of **Ax21** and MP.

For ($\vee$), we need to prove that:

$$\Gamma_1 \cup \Gamma_2 \vdash (\alpha_1 \vee \alpha_2) \to (\beta_1 \vee \beta_2) \tag{4.9}$$

$$\Gamma_1 \cup \Gamma_2 \vdash (\beta_1 \vee \beta_2) \to (\alpha_1 \vee \alpha_2) \tag{4.10}$$

$$\Gamma_1 \cup \Gamma_2 \vdash \sim(\alpha_1 \vee \alpha_2) \to \sim(\beta_1 \vee \beta_2) \tag{4.11}$$

$$\Gamma_1 \cup \Gamma_2 \vdash \sim(\beta_1 \vee \beta_2) \to \sim(\alpha_1 \vee \alpha_2) \tag{4.12}$$

For (4.9) and (4.10), we use **Ax6**, **Ax7**, **Ax8** and MP for inferring the conclusions. The item (4.12), follows the same line of reasoning from (4.11), so we will only show item (4.11).

1. $\sim\alpha_1 \to \sim\beta_1$ — Premise
2. $\sim\alpha_2 \to \sim\beta_2$ — Premise
3. $\overbrace{(\sim\alpha_1 \wedge \sim\alpha_2)}^{\varphi} \to \sim\alpha_1$ — Ax3
4. $(\sim\alpha_1 \wedge \sim\alpha_2) \to \sim\beta_1$ — Lemma 4, 1, 3
5. $(\sim\alpha_1 \wedge \sim\alpha_2) \to \sim\alpha_2$ — Ax4
6. $(\sim\alpha_1 \wedge \sim\alpha_2) \to \sim\beta_2$ — Lemma 4, 2, 5
7. $(\varphi \to \sim\beta_1) \to ((\varphi \to \sim\beta_2) \to (\varphi \to (\sim\beta_1 \wedge \sim\beta_2)))$ — Ax5
8. $(\varphi \to \sim\beta_2) \to (\varphi \to (\sim\beta_1 \wedge \sim\beta_2))$ — MP, 4, 7
9. $(\sim\alpha_1 \wedge \sim\alpha_2) \to (\sim\beta_1 \wedge \sim\beta_2)$ — MP, 6, 8
10. $\sim(\alpha_1 \vee \alpha_2) \to (\sim\alpha_1 \wedge \sim\alpha_2)$ — Ax9 ($\to$)
11. $\sim(\alpha_1 \vee \alpha_2) \to (\sim\beta_1 \wedge \sim\beta_2)$ — Lemma 4, 9, 10
12. $(\sim\beta_1 \wedge \sim\beta_2) \to \sim(\beta_1 \vee \beta_2)$ — Ax9 ($\leftarrow$)
13. $\sim(\alpha_1 \vee \alpha_2) \to \sim(\beta_1 \vee \beta_2)$ — Lemma 4, 10, 11

For $(\rightarrow)$, we need to prove that:

$$\Gamma_1 \cup \Gamma_2 \vdash (\alpha_1 \rightarrow \alpha_2) \rightarrow (\beta_1 \rightarrow \beta_2) \tag{4.13}$$

$$\Gamma_1 \cup \Gamma_2 \vdash (\beta_1 \rightarrow \beta_2) \rightarrow (\alpha_1 \rightarrow \alpha_2) \tag{4.14}$$

$$\Gamma_1 \cup \Gamma_2 \vdash \sim(\alpha_1 \rightarrow \alpha_2) \rightarrow \sim(\beta_1 \rightarrow \beta_2) \tag{4.15}$$

$$\Gamma_1 \cup \Gamma_2 \vdash \sim(\beta_1 \rightarrow \beta_2) \rightarrow \sim(\alpha_1 \rightarrow \alpha_2) \tag{4.16}$$

Lemma 4 is used in (4.13) and (4.14) for inferring the conclusions. The item (4.16), follows the same line of reasoning from (4.15), so we will only show item (4.15).

1. $\alpha_1 \rightarrow \beta_1$ Premise
2. $\sim\alpha_2 \rightarrow \sim\beta_2$ Premise
3. $(\alpha_1 \rightarrow \beta_1) \rightarrow (\sim\sim\alpha_1 \rightarrow \sim\sim\beta_1)$ Ax17
4. $\sim\sim\alpha_1 \rightarrow \sim\sim\beta_1$ MP, 1, 3
5. $(\sim\sim\alpha_1 \wedge \sim\sim\sim\alpha_2) \rightarrow \sim\sim\alpha_1$ Ax3
6. $\overbrace{(\sim\sim\alpha_1 \wedge \sim\sim\sim\alpha_2) \rightarrow \sim\sim\beta_1}^{\varphi}$ Lemma 4, 4, 5
7. $(\sim\alpha_2 \rightarrow \sim\beta_2) \rightarrow (\sim\sim\sim\alpha_2 \rightarrow \sim\sim\sim\beta_2)$ Ax17
8. $\sim\sim\sim\alpha_2 \rightarrow \sim\sim\sim\beta_2$ MP, 2, 7
9. $(\sim\sim\alpha_1 \wedge \sim\sim\sim\alpha_2) \rightarrow \sim\sim\sim\alpha_2$ Ax4
10. $\overbrace{(\sim\sim\alpha_1 \wedge \sim\sim\sim\alpha_2) \rightarrow \sim\sim\sim\beta_2}^{\psi}$ Lemma 4, 8, 9
11. $\varphi \rightarrow (\psi \rightarrow ((\sim\sim\alpha_1 \wedge \sim\sim\sim\alpha_2) \rightarrow (\sim\sim\beta_1 \wedge \sim\sim\sim\beta_2)))$ Ax5
12. $\psi \rightarrow ((\sim\sim\alpha_1 \wedge \sim\sim\sim\alpha_2) \rightarrow (\sim\sim\beta_1 \wedge \sim\sim\sim\beta_2))$ MP, 6, 11
13. $(\sim\sim\alpha_1 \wedge \sim\sim\sim\alpha_2) \rightarrow (\sim\sim\beta_1 \wedge \sim\sim\sim\beta_2)$ MP, 10, 12
14. $\sim\sim(\alpha_1 \wedge \sim\alpha_2) \rightarrow (\sim\sim\alpha_1 \wedge \sim\sim\sim\alpha_2)$ Ax14 $(\rightarrow)$
15. $\sim\sim(\alpha_1 \wedge \sim\alpha_2) \rightarrow (\sim\sim\beta_1 \wedge \sim\sim\sim\beta_2)$ Lemma 4, 13, 14
16. $(\sim\sim\beta_1 \wedge \sim\sim\sim\beta_2) \rightarrow \sim\sim(\beta_1 \wedge \sim\beta_2)$ Ax14 $(\leftarrow)$
17. $\sim\sim(\alpha_1 \wedge \sim\alpha_2) \rightarrow \sim\sim(\beta_1 \wedge \sim\beta_2)$ Lemma 4, 15, 16
18. $\sim(\alpha_1 \rightarrow \alpha_2) \rightarrow \sim\sim(\alpha_1 \wedge \sim\alpha_2)$ Ax10 $(\rightarrow)$
19. $\sim(\alpha_1 \rightarrow \alpha_2) \rightarrow \sim\sim(\beta_1 \wedge \sim\beta_2)$ Lemma 4, 17, 18
20. $\sim\sim(\beta_1 \wedge \sim\beta_2) \rightarrow \sim(\beta_1 \rightarrow \beta_2)$ Ax10 $(\leftarrow)$
21. $\sim(\alpha_1 \rightarrow \alpha_2) \rightarrow \sim(\beta_1 \rightarrow \beta_2)$ Lemma 4, 19, 20

□

Having proved that our calculus is algebraizable in the sense Blok and Pigozzi, we have a corresponding equivalent algebraic semantics $\mathtt{Alg}^*(\mathcal{L}_{\mathbf{QN4}})$, which satisfies the following equations and quasi-equations:

1. $E(p)$ for each $p \in \mathbf{Ax}$.
2. $E(\Delta(p, p))$.
3. $E(p)$ and $E(p \to q)$ implies $E(q)$.
4. $E(\Delta(p, q))$ implies $p = q$.

.

4.4 $\mathtt{Alg}^*(\mathcal{L}_{\mathbf{QN4}}) = \mathcal{V}_{\mathbf{QN4}}$

In order to prove that the class of algebras introduced in $\mathtt{Alg}^*(\mathcal{L}_{\mathbf{QN4}})$ is term-equivalent to the class of QN4-lattices, that is, $\mathtt{Alg}^*(\mathcal{L}_{\mathbf{QN4}}) = \mathcal{V}_{\mathbf{QN4}}$.

Proposition 8. $\mathtt{Alg}^*(\mathcal{L}_{\mathbf{QN4}}) \subseteq \mathcal{V}_{\mathbf{QN4}}$.

Proof. For proving **QN4a**, we need to show that the idempotent, commutative, absorption, associative and distributive laws hold for every $\mathbf{A} \in \mathtt{Alg}^*(\mathcal{L}_{\mathbf{QN4}})$.

1. Idempotent laws.

   For the law $x \wedge x = x$, we need to have that $(x \wedge x) \to x = |(x \wedge x) \to x|$, $x \to (x \wedge x) = |x \to (x \wedge x)|$, ${\sim}(x \wedge x) \to {\sim} x = |{\sim}(x \wedge x) \to {\sim} x|$ and ${\sim} x \to {\sim}(x \wedge x) = |{\sim} x \to {\sim}(x \wedge x)|$. In order to have these four equations in the algebra, we must prove in the logic the following four axioms:

   a) $(\alpha \wedge \alpha) \to \alpha$, this is an instantiation of **Ax3**.

   b) $\alpha \to (\alpha \wedge \alpha)$, shown using **Ax5**, Proposition 1 and MP.

   c) ${\sim}(\alpha \wedge \alpha) \to {\sim} \alpha$, shown using **Ax20**, Proposition 1 and MP.

   d) ${\sim} \alpha \to {\sim}(\alpha \wedge \alpha)$, this is an instantiation of **Ax18**.

   The same idea applies to $x \vee x = x$.

2. Commutative laws

   For the law $x \wedge y = y \wedge x$, we have:

   a) $(\alpha \wedge \beta) \to (\beta \wedge \alpha)$

1.	$((\alpha \wedge \beta) \to \beta) \to (((\alpha \wedge \beta) \to \alpha) \to ((\alpha \wedge \beta) \to (\beta \wedge \alpha)))$	Ax5
2.	$(\alpha \wedge \beta) \to \beta$	Ax4
3.	$((\alpha \wedge \beta) \to \alpha) \to ((\alpha \wedge \beta) \to (\beta \wedge \alpha))$	MP, 1, 2
4.	$(\alpha \wedge \beta) \to \alpha$	Ax3
5.	$(\alpha \wedge \beta) \to (\beta \wedge \alpha)$	MP, 3, 4

   b) $(\beta \wedge \alpha) \to (\alpha \wedge \beta)$, this is an instantiation of previous item.

   c) $\sim(\alpha \wedge \beta) \to \sim(\beta \wedge \alpha)$, this is **Ax19**.

   d) $\sim(\beta \wedge \alpha) \to \sim(\alpha \wedge \beta)$, this is an instantiation of **Ax19**.

   The same idea applies to $x \vee y = y \vee x$.

3. Absorption laws.

   For the law $x \wedge (x \vee y) = x$, we have:

   a) $(\alpha \wedge (\alpha \vee \beta)) \to \alpha$, this is an instantiation of **Ax3**.

   b) $\alpha \to (\alpha \wedge (\alpha \vee \beta))$

1.	$(\alpha \to \alpha) \to ((\alpha \to (\alpha \vee \beta)) \to (\alpha \to (\alpha \wedge (\alpha \vee \beta))))$	Ax5
2.	$\alpha \to \alpha$	Proposition 1
3.	$(\alpha \to (\alpha \vee \beta)) \to (\alpha \to (\alpha \wedge (\alpha \vee \beta)))$	MP, 1, 2
4.	$\alpha \to (\alpha \vee \beta)$	Ax6
5.	$\alpha \to (\alpha \wedge (\alpha \vee \beta))$	MP, 3, 4

   c) $\sim(\alpha \wedge (\alpha \vee \beta)) \to \sim\alpha$

1.	$\sim(\alpha \wedge (\alpha \vee \beta)) \to \sim((\alpha \wedge \alpha) \vee (\alpha \wedge \beta))$	Ax12 ($\to$)
2.	$\sim((\alpha \wedge \alpha) \vee (\alpha \wedge \beta)) \to (\sim(\alpha \wedge \alpha) \wedge \sim(\alpha \wedge \beta))$	Ax9 ($\to$)
3.	$\sim(\alpha \wedge (\alpha \vee \beta)) \to (\sim(\alpha \wedge \alpha) \wedge \sim(\alpha \wedge \beta))$	Lemma 4, 1, 2
4.	$(\sim(\alpha \wedge \alpha) \wedge \sim(\alpha \wedge \beta)) \to \sim(\alpha \wedge \alpha)$	Ax3
5.	$\sim(\alpha \wedge (\alpha \vee \beta)) \to \sim(\alpha \wedge \alpha)$	Lemma 4, 3, 4
6.	$(\sim\alpha \to \sim\alpha) \to (\sim(\alpha \wedge \alpha) \to \sim\alpha)$	Ax20
7.	$\sim\alpha \to \sim\alpha$	Proposition 1

8.	$\sim(\alpha \wedge \alpha) \to \sim\alpha$	MP, 6, 7
9.	$\sim(\alpha \wedge (\alpha \vee \beta)) \to \sim\alpha$	Lemma 4, 5, 8

d) $\sim\alpha \to \sim(\alpha \wedge (\alpha \vee \beta))$, this is an instantiation of **Ax18**.

The same idea applies to $x \vee (x \wedge y) = x$.

4. Associative laws.

For the law $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, we have:

a) $(\alpha \wedge (\beta \wedge \gamma)) \to ((\alpha \wedge \beta) \wedge \gamma)$

1.	$((\alpha \wedge (\beta \wedge \gamma)) \to (\alpha \wedge \beta)) \to ((\alpha \wedge (\beta \wedge \gamma)) \to \gamma) \to ((\alpha \wedge (\beta \wedge \gamma)) \to ((\alpha \wedge \beta) \wedge \gamma)))$	Ax5
2.	$(\alpha \wedge (\beta \wedge \gamma)) \to \alpha$	Ax3
3.	$(\alpha \wedge (\beta \wedge \gamma)) \to \beta \wedge \gamma$	Ax4
4.	$(\beta \wedge \gamma) \to \beta$	Ax3
5.	$(\alpha \wedge (\beta \wedge \gamma)) \to \beta$	Lemma 4, 3, 4
6.	$((\alpha \wedge (\beta \wedge \gamma)) \to \alpha) \to (((\alpha \wedge (\beta \wedge \gamma) \to \beta) \to (((\alpha \wedge (\beta \wedge \gamma)) \to (\alpha \wedge \beta)))$	Ax5
7.	$((\alpha \wedge (\beta \wedge \gamma) \to \beta) \to (((\alpha \wedge (\beta \wedge \gamma)) \to (\alpha \wedge \beta))$	MP, 2, 6
8.	$((\alpha \wedge (\beta \wedge \gamma)) \to (\alpha \wedge \beta)$	MP, 5, 7
9.	$(\alpha \wedge (\beta \wedge \gamma)) \to \gamma) \to ((\alpha \wedge (\beta \wedge \gamma)) \to ((\alpha \wedge \beta) \wedge \gamma))$	MP, 1, 8
10.	$(\beta \wedge \gamma) \to \gamma$	Ax4
11.	$(\alpha \wedge (\beta \wedge \gamma)) \to \gamma$	Lemma 4, 3, 10
12.	$(\alpha \wedge (\beta \wedge \gamma)) \to ((\alpha \wedge \beta) \wedge \gamma)$	MP, 9, 11

b) $((\alpha \wedge \beta) \wedge \gamma) \to (\alpha \wedge (\beta \wedge \gamma))$

1.	$(((\alpha \wedge \beta) \wedge \gamma) \to \alpha) \to (((\alpha \wedge \beta) \wedge \gamma) \to (\beta \wedge \gamma)) \to (((\alpha \wedge \beta) \wedge \gamma) \to (\alpha \wedge (\beta \wedge \gamma))))$	Ax5
2.	$((\alpha \wedge \beta) \wedge \gamma) \to (\alpha \wedge \beta)$	Ax3
3.	$(\alpha \wedge \beta) \to \alpha$	Ax3
4.	$((\alpha \wedge \beta) \wedge \gamma) \to \alpha$	Lemma 4, 2, 3
5.	$((\alpha \wedge \beta) \wedge \gamma) \to (\beta \wedge \gamma)) \to (((\alpha \wedge \beta) \wedge \gamma) \to (\alpha \wedge (\beta \wedge \gamma)))$	MP, 1, 4
6.	$((\alpha \wedge \beta) \wedge \gamma) \to \gamma$	Ax4
7.	$(\alpha \wedge \beta) \to \beta$	Ax4
8.	$((\alpha \wedge \beta) \wedge \gamma) \to \beta$	Lemma 4, 2, 7
9.	$(((\alpha \wedge \beta) \wedge \gamma) \to \beta) \to ((((\alpha \wedge \beta) \wedge \gamma) \to \gamma) \to (((\alpha \wedge \beta) \wedge \gamma) \to (\beta \wedge \gamma)))$	Ax5
10.	$((\alpha \wedge \beta) \wedge \gamma) \to \gamma) \to (((\alpha \wedge \beta) \wedge \gamma) \to (\beta \wedge \gamma))$	MP, 8, 9
11.	$((\alpha \wedge \beta) \wedge \gamma) \to (\beta \wedge \gamma)$	MP, 6, 10
12.	$((\alpha \wedge \beta) \wedge \gamma) \to (\alpha \wedge (\beta \wedge \gamma))$	MP 5, 11

c) $\sim(\alpha \wedge (\beta \wedge \gamma)) \to \sim((\alpha \wedge \beta) \wedge \gamma)$, this is **Ax11** ($\to$).

d) $\sim((\alpha \wedge \beta) \wedge \gamma) \to \sim(\alpha \wedge (\beta \wedge \gamma))$, this is **Ax11** ($\leftarrow$).

The same idea applies to $x \vee (y \vee z) = (x \vee y) \vee z$.

5. Distributive laws.

   Axioms **Ax1-Ax8** of $\mathcal{L}_{\mathbf{QN4}}$ are the axioms of the Positive Logic and it is known that the distributive law holds in this logic. Distributive law and **Ax11** give us the distributivity in the lattice.

Clearly, **QN4d** is axiom 10, **QN4e.1** is axiom 15, **QN4e.3** is axiom 9 and **QN4e.4** is axiom 14. For **QN4e.2**, that is, $\sim a = \sim\sim\sim a$, we have that $\sim\sim\sim a \leq \sim a$ by axiom 22. It remains to prove that $\sim a \leq \sim\sim\sim a$, this is an instantiation of axiom 15. Instead proving **QN4b** and **QN4c**, we can prove that $\mathtt{Alg}^*(\mathcal{L}_{\mathbf{QN4}})$ satisfies the equations of Proposition 7 and these proves are straightforward.

□

Proposition 9. $\mathcal{V}_{\mathbf{QN4}} \subseteq \mathtt{Alg}^*(\mathcal{L}_{\mathbf{QN4}})$.

Proof. Let $\mathbf{A} \in \mathbf{QN4}$, and let $a, b, c \in A$ be generic elements. By Theorem 10, we assume that $\mathbf{A}$ is a twist-structure, and from now on we also denote $a = \langle a_1, a_2\rangle$, $b = \langle b_1, b_2\rangle$ and $c = \langle c_1, c_2\rangle$. Note that, proving $E(a)$ for a given element a is equivalent to showing that $\pi_1(a) = 1$. We shall use this observation without further notice throughout the proof.

It is very easy to see that the twist-structure definitions, together with the Brouwerian algebra properties, entail that $\pi_1(\mathbf{Axn}) = 1$ for $1 \leq n \leq 8$. In the case of $E(a \leftrightarrow b)$, it is equivalent to prove that $\pi_1(a \leftrightarrow b) = 1$, which in turn is equivalent to proving $\pi_1(a) = \pi_1(b)$, this is, $a_1 = b_1$. So,

- $E(\sim(a \vee b) \leftrightarrow (\sim a \wedge \sim b))$

  On the one hand, $\pi_1[\sim(a \vee b)] = \pi_1[\sim(\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle)] = \pi_1[\sim\langle a_1 \vee b_1, a_2 \wedge b_2\rangle] = \pi_1[\langle a_2 \wedge b_2, \Box(a_1 \vee b_1)\rangle] = a_2 \wedge b_2$.

  On the other hand, $\pi_1[\sim a \wedge \sim b] = \pi_1[\sim\langle a_1, a_2\rangle \wedge \sim\langle b_1, b_2\rangle] = \pi_1[\langle a_2, \Box a_1\rangle \wedge \langle b_2, \Box b_1\rangle] = \pi_1[\langle a_2 \wedge b_2, \Box(\Box a_1 \vee \Box b_1)\rangle] = a_2 \wedge b_2$.

- $E(\sim(a \to b) \leftrightarrow \sim\sim(a \wedge \sim b))$

  On the one hand, $\pi_1[\sim(a \to b)] = \pi_1[\sim(\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle)] = \pi_1[\sim\langle a_1 \to b_1, \Box a_1 \wedge b_2\rangle] = \pi_1[\langle \Box a_1 \wedge b_2, \Box(a_1 \to b_1)\rangle] = \Box a_1 \wedge b_2$.

  On the other hand, $\pi_1[\sim\sim(a \wedge \sim b)] = \pi_1[\sim\sim(\langle a_1, a_2\rangle \wedge \sim\langle b_1, b_2\rangle)] = \pi_1[\sim\sim(\langle a_1, a_2\rangle \wedge \langle b_2, \Box b_1\rangle)] = \pi_1[\sim\sim\langle a_1 \wedge b_2, \Box(a_2 \vee \Box b_1)\rangle] = \pi_1[\sim\langle \Box(a_2 \vee \Box b_1), \Box(a_1 \wedge b_2)\rangle] = \pi_1[\langle \Box(a_1 \wedge b_2), \Box(\Box(a_2 \vee \Box b_1))\rangle] = \Box(a_1 \wedge b_2) = \Box a_1 \wedge \Box b_2 = \Box a_1 \wedge b_2$.

- $E(\sim(a \wedge (b \wedge c)) \leftrightarrow \sim((a \wedge b) \wedge c))$

  On the one hand, $\pi_1[\sim(a \wedge (b \wedge c))] = \pi_1[\sim(\langle a_1, a_2\rangle \wedge (\langle b_1, b_2\rangle \wedge \langle c_1, c_2\rangle))] = \pi_1[\sim(\langle a_1, a_2\rangle \wedge \langle b_1 \wedge c_1, \Box(b_2 \vee c_2)\rangle)] = \pi_1[\sim\langle a_1 \wedge (b_1 \wedge c_1), \Box(a_2 \vee \Box(b_2 \vee c_2))\rangle] = \pi_1[\langle \Box(a_2 \vee \Box(b_2 \vee c_2)), \Box(a_1 \wedge (b_1 \wedge c_1))\rangle] = \Box(a_2 \vee \Box(b_2 \vee c_2)) = \Box(a_2 \vee (b_2 \vee c_2)) = a_2 \vee (b_2 \vee c_2) = a_2 \vee b_2 \vee c_2$.

  On the other hand, $\pi_1[\sim((a \wedge b) \wedge c)] = \pi_1[\sim((\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle) \wedge \langle c_1, c_2\rangle)] = \pi_1[\sim(\langle a_1 \wedge b_1, \Box(a_2 \vee b_2)\rangle \wedge \langle c_1, c_2\rangle)] = \pi_1[\sim\langle a_1 \wedge b_1 \wedge c_1, \Box(\Box(a_2 \vee b_2) \vee c_2)\rangle] = \pi_1[\langle \Box(\Box(a_2 \vee b_2) \vee c_2), \Box((a_1 \wedge b_1 \wedge c_1))\rangle] = \Box(\Box(a_2 \vee b_2) \vee c_2) = \Box((a_2 \vee b_2) \vee c_2) = (a_2 \vee b_2) \vee c_2 = a_2 \vee b_2 \vee c_2$.

- $E(\sim(a \wedge (b \vee c)) \leftrightarrow \sim((a \wedge b) \vee (a \wedge c)))$

  On the one hand, $\pi_1[\sim(a \wedge (b \vee c))] = \pi_1[\sim(\langle a_1, a_2\rangle \wedge (\langle b_1, b_2\rangle \vee \langle c_1, c_2\rangle))]\pi_1[\sim(\langle a_1, a_2\rangle \wedge \langle b_1 \vee c_1, b_2 \wedge c_2\rangle)] = \pi_1[\sim\langle a_1 \wedge (b_1 \vee c_1), \Box(a_2 \vee (b_2 \wedge c_2))\rangle] = \pi_1[\langle \Box(a_2 \vee (b_2 \wedge c_2)), \Box(a_1 \wedge (b_1 \vee c_1))\rangle] = \Box(a_2 \vee (b_2 \wedge c_2)) = a_2 \vee (b_2 \wedge c_2)$.

  On the other hand, $\pi_1[\sim((a \wedge b) \vee (a \wedge c))] = \pi_1[\sim((\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle) \vee (\langle a_1, a_2\rangle \wedge \langle c_1, c_2\rangle))] = \pi_1[\sim(\langle a_1 \wedge b_1, \Box(a_2 \vee b_2)\rangle \vee \langle a_1 \wedge c_1, \Box(a_2 \vee c_2)\rangle)] = \pi_1[\sim\langle (a_1 \wedge b_1) \vee (a_1 \wedge c_1), \Box(a_2 \vee b_2) \wedge \Box(a_2 \vee c_2)\rangle] = \pi_1[\langle \Box(a_2 \vee b_2) \wedge \Box(a_2 \vee c_2), \Box((a_1 \wedge b_1) \vee (a_1 \wedge c_1))\rangle] = \Box(a_2 \vee b_2) \wedge \Box(a_2 \vee c_2) = (a_2 \vee b_2) \wedge (a_2 \vee c_2) = a_2 \vee (b_2 \wedge c_2)$.

- $E(\sim(a \vee (b \wedge c)) \leftrightarrow \sim((a \vee b) \wedge (a \vee c)))$

  On the one hand, $\pi_1[\sim(a \vee (b \wedge c))] = \pi_1[\sim(\langle a_1, a_2\rangle \vee (\langle b_1, b_2\rangle \wedge \langle c_1, c_2\rangle))] = \pi_1[\sim(\langle a_1, a_2\rangle \vee \langle b_1 \wedge c_1, \Box(b_2 \vee c_2)\rangle)] = \pi_1[\sim\langle a_1 \vee (b_1 \wedge c_1), a_2 \wedge \Box(b_2 \vee c_2)\rangle] =$

$\pi_1[\langle a_2 \wedge \Box(b_2 \vee c_2), \Box(a_1 \vee (b_1 \wedge c_1))\rangle] = a_2 \wedge \Box(b_2 \vee c_2) = a_2 \wedge (b_2 \vee c_2)$.

On the other hand, $\pi_1[\sim((a \vee b) \wedge (a \vee c))] = \pi_1[\sim((\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle) \wedge (\langle a_1, a_2\rangle \vee \langle c_1, c_2\rangle))] = \pi_1[\sim(\langle a_1 \vee b_1, a_2 \wedge b_2\rangle \wedge \langle a_1 \vee c_1, a_2 \wedge c_2\rangle)] = \pi_1[\sim\langle(a_1 \vee b_1) \wedge (a_1 \vee c_1), \Box((a_2 \wedge b_2) \vee (a_2 \wedge c_2))\rangle] = \pi_1[\langle\Box((a_2 \wedge b_2) \vee (a_2 \wedge c_2)), \Box((a_1 \vee b_1) \wedge (a_1 \vee c_1))\rangle] = \Box((a_2 \wedge b_2) \vee (a_2 \wedge c_2)) = (a_2 \wedge b_2) \vee (a_2 \wedge c_2) = a_2 \wedge (b_2 \vee c_2)$.

- $E(\sim\sim(a \wedge b) \leftrightarrow (\sim\sim a \wedge \sim\sim b))$

  On the one hand, $\pi_1[\sim\sim(a \wedge b)] = \pi_1[\sim\sim(\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle)] = \pi_1[\sim\sim\langle a_1 \wedge b_1, \Box(a_2 \vee b_2)\rangle] = \pi_1[\sim\langle\Box(a_2 \vee b_2), \Box(a_1 \wedge b_1)\rangle] = \pi_1[\langle\Box(a_1 \wedge b_1), \Box(\Box(a_2 \vee b_2))\rangle] = \Box(a_1 \wedge b_1) = \Box a_1 \wedge \Box b_1$.

  On the other hand, $\pi_1[\sim\sim a \wedge \sim\sim b] = \pi_1[\sim\sim\langle a_1, a_2\rangle \wedge \sim\sim\langle b_1, b_2\rangle] = \pi_1[\sim\langle a_2, \Box a_1\rangle \wedge \sim\langle b_2, \Box b_1\rangle] = \pi_1[\langle\Box a_1, \Box a_2\rangle \wedge \langle\Box b_1, \Box b_2\rangle] = \pi_1[\langle\Box a_1 \wedge \Box b_1, \Box(\Box a_2 \vee \Box b_2)\rangle] = \Box a_1 \wedge \Box b_1$.

  Already in case of $E(a \to b)$ saying this is equivalent to proving that $\pi_1(a) \leq \pi_1(b)$, this is, $a_1 \leq b_1$. So,

- $E(a \to \sim\sim a)$

  On the one hand, $\pi_1[a] = \pi_1[\langle a_1, a_2\rangle] = a_1$.

  On the other hand, $\pi_1[\sim\sim a] = \pi_1[\sim\sim\langle a_1, a_2\rangle] = \pi_1[\sim\langle a_2, \Box a_1\rangle] = \pi_1[\langle\Box a_1, \Box a_2\rangle] = \Box a_1$.

- $E(a \to (\sim a \to \sim(a \to a)))$

  On the one hand, $\pi_1[a] = \pi_1[\langle a_1, a_2\rangle] = a_1$.

  On the other hand, $\pi_1[\sim a \to \sim(a \to a)] = \pi_1[\sim\langle a_1, a_2\rangle \to \sim(\langle a_1, a_2\rangle \to \langle a_1, a_2\rangle)] = \pi_1[\langle a_2, \Box a_1\rangle \to \sim\langle a_1 \to a_1, \Box a_1 \wedge a_2\rangle] = \pi_1[\langle a_2, \Box a_1\rangle \to \langle\Box a_1 \wedge a_2, \Box(a_1 \to a_1)\rangle = \pi_1[\langle a_2 \to (\Box a_1 \wedge a_2), \Box a_2 \wedge \Box(a_1 \to a_1)] = a_2 \to (\Box a_1 \wedge a_2)$.

- $E((a \to b) \to (\sim\sim a \to \sim\sim b))$

  On the one hand, $\pi_1[a \to b] = \pi_1[\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle] = \pi_1[\langle a_1 \to b_1, \Box a_1 \wedge b_2\rangle] = a_1 \to b_1$.

On the other hand, $\pi_1[\sim\sim a \to \sim\sim b] = \pi_1[\sim\sim\langle a_1, a_2\rangle \to \sim\sim\langle b_1, b_2\rangle] = \pi_1[\sim\langle a_2, \Box a_1\rangle \to \sim\langle b_2, \Box b_1\rangle] = \pi_1[\langle \Box a_1, \Box a_2\rangle \to \langle \Box b_1, \Box b_2\rangle] = \pi_1[\langle \Box a_1 \to \Box b_1, \Box\Box a_1 \wedge \Box b_2\rangle] = \Box a_1 \to \Box b_1$.

- $E(\sim a \to \sim(a \wedge b))$

  On the one hand, $\pi_1[\sim a] = \pi_1[\sim\langle a_1, a_2\rangle] = \pi_1[\langle a_2, \Box a_1\rangle] = a_2$.

  On the other hand, $\pi_1[\sim(a \wedge b)] = \pi_1[\sim(\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle)] = \pi_1[\sim\langle a_1 \wedge b_1, \Box(a_2 \vee b_2)\rangle] = \pi_1[\langle \Box(a_2 \vee b_2), \Box(a_1 \wedge b_1)\rangle] = \Box(a_2 \vee b_2) = a_2 \vee b_2$.

- $E(\sim(a \wedge b) \to \sim(b \wedge a))$

  On the one hand, $\pi_1[\sim(a \wedge b)] = \pi_1[\sim(\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle)] = \pi_1[\sim\langle a_1 \wedge b_1, \Box(a_2 \vee b_2)\rangle] = \pi_1[\langle \Box(a_2 \vee b_2), \Box(a_1 \wedge b_1)\rangle] = \Box(a_2 \vee b_2) = a_2 \vee b_2$.

  On the other hand, $\pi_1[\sim(b \wedge a)] = \pi_1[\sim(\langle b_1, b_2\rangle \wedge \langle a_1, a_2\rangle)] = \pi_1[\sim\langle b_1 \wedge a_1, \Box(b_2 \vee a_2)\rangle] = \pi_1[\langle \Box(b_2 \vee a_2), \Box(b_1 \wedge a_1)\rangle] = \Box(b_2 \vee a_2) = b_2 \vee a_2$.

- $E((\sim a \to \sim b) \to (\sim(a \wedge b) \to \sim b))$

  On the one hand, $\pi_1[\sim a \to \sim b] = \pi_1[\sim\langle a_1, a_2\rangle \to \sim\langle b_1, b_2\rangle] = \pi_1[\langle a_2, \Box a_1\rangle \to \langle b_2, \Box b_1\rangle] = \pi_1[\langle a_2 \to b_2, \Box a_2 \wedge \Box b_1\rangle] = a_2 \to b_2$.

  On the other hand, $\pi_1[\sim(a \wedge b) \to \sim b] = \pi_1[\sim(\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle) \to \sim\langle b_1, b_2\rangle] = \pi_1[\sim\langle a_1 \wedge b_1, \Box(a_2 \vee b_2)\rangle \to \langle b_2, \Box b_1\rangle] = \pi_1[\langle \Box(a_2 \vee b_2), \Box(a_1 \wedge b_1)\rangle \to \langle b_2, \Box b_1\rangle] = \pi_1[\langle \Box(a_2 \vee b_2) \to b_2, \Box\Box(a_2 \vee b_2) \wedge \Box b_1\rangle] = \Box(a_2 \vee b_2) \to b_2 = (a_2 \vee b_2) \to b_2$.

- $E((\sim a \to \sim b) \to ((\sim c \to \sim \theta) \to (\sim(a \wedge c) \to \sim(b \wedge \theta))))$

  On the one hand, $\pi_1[\sim a \to \sim b] = \pi_1[\sim\langle a_1, a_2\rangle \to \sim\langle b_1, b_2\rangle] = \pi_1[\langle a_2, \Box a_1\rangle \to \langle b_2, \Box b_1\rangle] = \pi_1[\langle a_2 \to b_2, \Box a_2 \wedge \Box b_1\rangle] = a_2 \to b_2$.

  On the other hand, $\pi_1[(\sim c \to \sim d) \to (\sim(a \wedge c) \to \sim(b \wedge d))] = \pi_1[(\sim\langle c_1, c_2\rangle \to \sim\langle d_1, d_2\rangle) \to (\sim(\langle a_1, a_2\rangle \wedge \langle c_1, c_2\rangle) \to \sim(\langle b_1, b_2\rangle \wedge \langle d_1, d_2\rangle))] = \pi_1[(\langle c_2, \Box c_1\rangle \to \langle d_2, \Box d_1\rangle) \to (\sim\langle a_1 \wedge c_1, \Box(a_2 \vee c_2)\rangle \to \sim\langle b_1 \wedge d_1, \Box(b_2 \vee d_2)\rangle)] = \pi_1[\langle c_2 \to d_2, \Box c_2 \wedge \Box d_1\rangle \to (\langle \Box(a_2 \vee c_2), \Box(a_1 \wedge c_1)\rangle \to \langle \Box(b_2 \vee d_2), \Box(b_1 \wedge d_1)\rangle)] = \pi_1[\langle c_2 \to d_2, \Box c_2 \wedge \Box d_1\rangle \to \langle \Box(a_2 \vee c_2) \to \Box(b_2 \vee d_2), \Box\Box(a_2 \vee c_2) \wedge \Box(b_1 \wedge$

$d_1))\rangle] = \pi_1[\langle ((c_2 \to d_2) \to (\Box(a_2 \vee c_2) \to \Box(b_2 \vee d_2)), \Box(c_2 \to d_2) \wedge (\Box\Box(a_2 \vee c_2) \wedge \Box(b_1 \wedge d_1)))\rangle] = (c_2 \to d_2) \to (\Box(a_2 \vee c_2) \to \Box(b_2 \vee d_2)) = (c_2 \to d_2) \to ((a_2 \vee c_2) \to (b_2 \vee d_2))$.

- $E({\sim}{\sim}{\sim} a \to {\sim} a)$

  On the one hand, $\pi_1[{\sim}{\sim}{\sim} a] = \pi_1[{\sim}{\sim}{\sim}\langle a_1, a_2\rangle] = \pi_1[{\sim}{\sim}\langle a_2, \Box a_1\rangle] = \pi_1[{\sim}\langle \Box a_1, \Box a_2\rangle] = \pi_1[\langle \Box a_2, \Box\Box a_1\rangle] = \Box a_2 = a_2$.

  On the other hand, $\pi_1[{\sim} a] = \pi_1[{\sim}\langle a_1, a_2\rangle] = \pi_1[\langle a_2, \Box a_1\rangle] = a_2$.

We have to prove that $a \to a = (a \to a) \to (a \to a)$ and that ${\sim} a \to {\sim} a = ({\sim} a \to {\sim} a) \to ({\sim} a \to {\sim} a)$. Taking $|x| = y = (a \to a)$ in Proposition 7.1, we have that $(a \to a) \to (a \to a) = a \to a$, that is what we wanted to prove. The same idea for negation.

We want to prove that if $a \to b = |a \to b|$, $b \to a = |b \to a|$, ${\sim} a \to {\sim} a = |{\sim} a \to {\sim} b|$, ${\sim} b \to {\sim} a = |{\sim} b \to {\sim} a|$, then $a = b$. As $a \to b = |a \to b|$ and ${\sim} b \to {\sim} a = |{\sim} b \to {\sim} a|$, we have $a \preceq b$ and ${\sim} b \preceq {\sim} a$ and therefore by **QN4c** we conclude that $a \leq b$. We also have that $b \to a = |b \to a|$ and ${\sim} a \to {\sim} b = |{\sim} a \to {\sim} b|$ and therefore $b \preceq a$ and ${\sim} a \preceq {\sim} b$ and again by **QN4c** we conclude that $b \leq a$. As $a \leq b$ and $b \leq a$ we have $a = b$ and this is what we wanted to prove.

We have to prove that if $a = a \to a$, $a \to b = (a \to b) \to (a \to b)$ then $b = b \to b$. Again, using Proposition 7.1, taking $|x| = a \to a$ and $y = b$, we have that $(a \to a) \to b = b$, but we have that $a \to a = a$ and therefore $a \to b = b$, but as $a \to b = (a \to b) \to (a \to b)$ and $a \to b = b$ we have that $b = b \to b$. □

Corollary 1. The class of **QN4**-lattices and the class of $\mathtt{Alg}^*(\mathcal{L}_{\mathbf{QN4}})$-algebras coincide.

5 Fragments of $\mathcal{QNL}$

In this chapter we turn our attention to the fragment of quasi-Nelson logic that contain two "substructural" connectives, namely: the strong conjunction ($*$) and the strong implication ($\Rightarrow$), which together form a residuated pair over any quasi-Nelson algebra (viewed as residuated lattices).

The main question which we will address is whether the algebraic semantics of a given fragment of quasi-Nelson logic (i.e. the corresponding class of subreducts of quasi-Nelson algebras) can be axiomatized by means of equations or quasi-equations. Our main mathematical tool in this investigation will be the twist-algebra representation, which will allow us to establish a bridge between the subreducts of quasi-Nelson algebras and more well-known subreducts of Heyting algebras. For ease of reference, the classes of subreducts of quasi-Nelson algebras that have been characterized up to now are shown in Table 5.1.

It should be noted that some of the above-mentioned subreducts of quasi-Nelson algebras (namely, quasi-Nelson monoids, quasi-Kleene algebras with weak pseudo-complement and quasi-Kleene algebras) are not BP-algebraizable. However, quasi-Nelson implication algebras, quasi-Nelson pocrims and quasi-Nelson semihoops are BP-algebraizable.

Nascimento and Rivieccio in [17] began to study the $\{\sim, \to\}$-fragment. Now, we will focus on the study of the fragment $\{\sim, *, \Rightarrow\}$ and $\{\sim, *, \Rightarrow, \wedge\}$ of quasi-Nelson logic, respectively in the sections 5.1 and 5.2.

Operations	Subreducts of QNA
$\sim, \to$ $[0, 1, \neg]$	quasi-Nelson implication algebras
$\sim, *$ $[0, 1, \neg]$	quasi-Nelson monoids
$\sim, *, \to$ $[0, 1, \neg, \Rightarrow]$	quasi-Nelson pocrims
$\sim, \wedge, \to$ $[0, 1, \neg, *, \Rightarrow]$	quasi-Nelson semihoops
$\sim, \neg, \wedge, \vee$ $[0, 1]$	quasi-Kleene algebras with weak pseudo-complement
$0, \sim, \wedge, \vee$ $[1]$	quasi-Kleene algebras

Table 5.1: Subreducts of quasi-Nelson algebras characterized so far

For the next sections, we employ the following abbreviations:

$$\begin{cases} 1 := x \to x \\ 0 := \sim(x \to x) \\ |x| := x \to x \\ x \equiv y := x \to y = y \to x = 1 \\ x \odot y := \sim(x \to \sim y) \\ x \oplus y := \sim(\sim x \wedge \sim y) \\ q(x, y, z) := (x \to y) \to ((y \to x) \to ((\sim x \to \sim y) \to ((\sim y \to \sim x) \to z))) \end{cases}$$

Following standard notation on residuated lattices, given a natural number n, we define the term: $x^n := \underbrace{x * \ldots * x}_{n \text{ times}}$, where we set $x^0 := 1$ and $x^1 := x$. We say that the operation $*$ is $(n+1)$-potent when the equation $x^n = x^{n+1}$ is satisfied.

5.1 $\{\sim, *, \Rightarrow\}$-fragment

We will begin our section with definitions and important results for the understanding of the study of the $\{\sim, *, \Rightarrow\}$-fragment of the quasi-Nelson logic.

Definition 47 ([23], Def. 3.10)**.** A 3**-potent commutative monoid** is an algebra $\mathbf{M} = \langle M;\ *, 1\rangle$ of type $\langle 2, 0\rangle$ such that:

1. $\mathbf{M} = \langle M;\ *, 1\rangle$ is a commutative monoid.
2. $\mathbf{M} \models x^2 = x^3$.

Definition 48 ([23], Def. 4.1)**.** An algebra $\mathbf{A} = \langle A;\ \rightarrow, \sim, 0, 1\rangle$ of type $\langle 2, 1, 0, 0\rangle$ is a **quasi-Nelson implication algebra** (QNI-algebra) if the following equations are satisfied:

QNI.1 $\sim 1 = 0$ and $\sim 0 = 1$.

QNI.2 $1 \rightarrow x = x$.

QNI.3 $x \rightarrow (y \rightarrow x) = x \rightarrow x = 0 \rightarrow x = 1$.

QNI.4 $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$.

QNI.5 $(x \rightarrow y) \rightarrow (\sim\sim x \rightarrow \sim\sim y) = 1$.

QNI.6 $\sim x = \sim\sim\sim x$.

QNI.7 $q(x, y, x) = q(x, y, y)$.

QNI.8 $(x \odot y) \rightarrow z = \sim\sim x \rightarrow (\sim\sim y \rightarrow z)$.

QNI.9 $x \odot y \equiv y \odot x$.

QNI.10 $x \odot (y \odot z) \equiv (x \odot y) \odot z$.

QNI.11 $x \odot (x \rightarrow y) \equiv x \odot y$.

QNI.12 $\sim(x \rightarrow y) \equiv \sim\sim x \odot \sim y$.

QNI.13 $\sim x \rightarrow \sim y \equiv \sim x \rightarrow (\sim x \odot \sim y)$.

QNI.14 $(\sim\sim x \rightarrow \sim\sim y) \odot (\sim\sim x \rightarrow \sim\sim z) \equiv \sim\sim x \rightarrow (y \odot z)$.

The variety of QNI-algebras will be henceforth denoted by $\mathcal{V}_{\mathbf{QNI}}$.

Definition 49 ([23])**.** A structure $\langle P;\ \leq, *, 1\rangle$ of type $\langle 2, 0\rangle$ is called a **pomonoid** whenever:

(i) $\langle P;\ \leq\rangle$ is a partially ordered set having 1 as top element.

(ii) $\langle P;\ *, 1\rangle$ is a commutative monoid.

(iii) The order $\leq$ is compatible with the monoid operation, that is, $x \leq z$ and $y \leq w$ entail $x * y \leq z * w$.

Definition 50 ([23])**.** A **pocrim** (partially ordered commutative residuated integral monoid) is a structure $\langle P;\ \leq, \Rightarrow, *, 1\rangle$ of type $\langle 2, 1, 0\rangle$ such that:

(i) $\langle P;\ \leq, *, 1\rangle$ is a pomonoid.

(ii) The pair $(*, \Rightarrow)$ is residuated, that is, $x * y \leq z$ if and only if $x \leq y \Rightarrow z$.

Definition 51 ([23], Def. 4.9)**.** An algebra $\mathbf{A} = \langle A;\ \to, *, \sim, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ is a **quasi-Nelson pocrim (QNP)** whenever:

(QNPa) $\langle A;\ \to, \sim, 0, 1\rangle$ is a QNI-algebra.

(QNPb) $\langle A;\ *, 1\rangle$ is a 3-potent commutative monoid.

(QNPc) For all $x,\ y \in A$, we have:

(QNPc.1) $(x * y) \to z = x \to (y \to z)$.

(QNPc.2) $x \to (y * z) \equiv (x \to y) * (x \to z)$.

(QNPc.3) $\sim(x * y) \equiv (x \to \sim y) * (y \to \sim x)$.

(QNPc.4) $\sim(x \to y) \equiv \sim\sim x * \sim y$.

The variety of **QNP** will be henceforth denoted by $\mathcal{V}_{\mathbf{QNP}}$.

We now proceed to show ([23]), that every quasi-Nelson pocrim may be represented as a twist-algebra over an implicative semilattice enriched with a nucleus operator.

Definition 52 ([23], Def. 4.11)**.** A **bounded implicative semilattice with a nucleus** is an algebra $\mathbf{S} = \langle S;\ \to, \wedge, \Box, 0, 1\rangle$ such that:

1. $\langle S;\ \to, \wedge, 0, 1\rangle$ is a bounded implicative semilattice.
2. $\Box$ is a nucleus on the bounded Hilbert algebra reduct $\langle S;\ \to, 0, 1\rangle$.

Lemma 5 ([23], Lem. 4.13)**.** For every $\mathbf{A} = \langle A;\ \to, *, \sim, 0, 1\rangle \in \mathbf{QNP}$, the relation $\equiv$ is compatible with $*$ and the quotient $\mathbf{A}_{\bowtie} := \langle A/\equiv;\ \to, *, \Box, 0, 1\rangle$ is a bounded implicative semilattice with a nucleus given by $\Box[a] := [\sim\sim a]$ for all $a \in A$.

Definition 53 ([23], Def. 4.14)**.** Let $\mathbf{S} = \langle S;\ \to, \wedge, \Box, 0, 1\rangle$ be a bounded implicative semilattice with a nucleus. Define the algebra $\mathbf{S}^{\bowtie} = \langle S^{\bowtie};\ \to, *, \sim, 0, 1\rangle$ with universe: $S^{\bowtie} := \{\langle a_1, a_2\rangle \in S \times S : a_2 = \Box a_2, a_1 \wedge a_2 = 0\}$ and operations given, for all $\langle a_1, a_2\rangle$, $\langle b_1, b_2\rangle \in S \times S$ by:

$$0 := \langle 0, 1\rangle$$

$$1 := \langle 1, 0\rangle$$

$$\sim\langle a_1, a_2\rangle := \langle a_2, \Box a_1\rangle$$

$$\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle := \langle a_1 \to b_1, \Box a_1 \wedge b_2\rangle$$

$$\langle a_1, a_2\rangle * \langle b_1, b_2\rangle := \langle a_1 \wedge b_1, (a_1 \to b_2) \wedge (b_1 \to a_2)\rangle$$

A **QNP twist-algebra** over **S** is any subalgebra $\mathbf{A} \leq \mathbf{S}^{\bowtie}$ satisfying $\pi_1[A] = S$.

Theorem 12. [[23], Thm. 4.16] Every $\mathbf{A} \in \mathbf{QNP}$ is isomorphic to a QNP twist-algebra over the implicative semilattice with a nucleus $\mathbf{A}_{\bowtie}$ through the map $\iota : A \to A_{\bowtie} \times A_{\bowtie}$ given by $\iota(a) := \langle [a], [\sim a]\rangle$ for all $a \in A$.

5.1.1 A Hilbert-style calculus

In this subsection we introduce a Hilbert-style calculus that determines a logic, henceforth denoted by $\mathcal{L}_{\mathbf{QNP}}$. Our aim is to show that $\mathcal{L}_{\mathbf{QNP}}$ is algebraizable, and that its equivalent algebraic semantics is precisely the variety $\mathcal{V}_{\mathbf{QNP}}$.

The Hilbert-system for $\mathcal{L}_{\mathbf{QNP}}$ consists of the following axiom schemes together with the single inference rule of *modus ponens* (MP): $\alpha, \alpha \to \beta \vdash \beta$.

Ax1 $\alpha \to (\beta \to \alpha)$

Ax2 $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

Ax3 $\sim\sim\sim\alpha \to \sim\alpha$

Ax4 $(\alpha \to \beta) \to (\sim\sim\alpha \to \sim\sim\beta)$

Ax5 $\alpha \to \sim\sim\alpha$

Ax6 $(\alpha \odot (\alpha \to \beta)) \to (\alpha \odot \beta)$

Ax7 $\sim\sim\alpha \to (\sim\beta \to \sim(\alpha \to \beta))$

Ax8 $\sim(\alpha \to \beta) \to \sim\beta$

Ax9 $\sim(\alpha \to \beta) \to \sim\sim\alpha$

Ax10 $\sim(\alpha \to \alpha) \to \beta$

Ax11 $(\alpha * \beta) \to \alpha$

Ax12 $(\alpha * \beta) \to \beta$

Ax13 $\alpha \to (\beta \to (\alpha * \beta))$

Ax14 $\sim(\alpha * \beta) \leftrightarrow ((\alpha \to \sim\beta) * (\beta \to \sim\alpha))$

Ax15 $(\alpha \to (\beta * \gamma)) \leftrightarrow ((\alpha \to \beta) * (\alpha \to \gamma))$

Ax16 $\sim(\alpha * (\beta * \gamma)) \leftrightarrow \sim((\alpha * \beta) * \gamma))$

Ax17 $\sim((\alpha * \beta) \to \gamma) \to \sim(\alpha \to (\beta \to \gamma))$

Axioms **Ax1-Ax10** together with modus ponens constitute an axiomatization of the $\mathcal{L}_{\mathbf{QNI}}$. We started with them and choose between the axioms of $\mathcal{QNL}$ [14] the ones that were sound with respect to **QNP**, then we added axioms in order to prove that our calculus is algebraizable and that its equivalent algebraic semantics is the class of $\mathcal{V}_{\mathbf{QNP}}$ as defined in Definition 51.

Remark 13. The Deduction Theorem holds for $\mathcal{L}_{\mathbf{QNP}}$.

5.1.2 $\mathcal{L}_{\mathbf{QNP}}$ is BP-Algebraizable

In this subsection we prove that the calculus introduced in the previous subsection is algebraizable in sense of Blok and Pigozzi. Using this result, we will axiomatize the equivalent algebraic semantics of $\mathcal{L}_{\mathbf{QNP}}$ via the algorithm in Theorem 8 and show that is term-equivalent to the variety $\mathcal{V}_{\mathbf{QNP}}$.

Theorem 13. $\mathcal{L}_{\mathbf{QNP}}$ is BP-algebraizable with $E(\alpha) := \{\alpha = \alpha \to \alpha\}$ and $\Delta(\alpha, \beta) := \{\alpha \to \beta, \beta \to \alpha, \sim\alpha \to \sim\beta, \sim\beta \to \sim\alpha\}$.

Proof. By Theorem 7, we have prove (**Ref**), (**MP**), (**Alg**) and (**Cong**). The first three follow the same reasoning of Theorem 11. As to (**Cong**), we need to prove it for each connective $\lambda \in \{\to, *, \sim\}$.

For $(\sim)$, follow the same reasoning of Theorem 11.

Now consider the sets: $\Gamma_1 = \{\alpha_1 \to \beta_1, \beta_1 \to \alpha_1, \sim\alpha_1 \to \sim\beta_1, \sim\beta_1 \to \sim\alpha_1\}$ and $\Gamma_2 = \{\alpha_2 \to \beta_2, \beta_2 \to \alpha_2, \sim\alpha_2 \to \sim\beta_2, \sim\beta_2 \to \sim\alpha_2\}$.

For $(\to)$ and $(*)$, we need to prove that:

$$\Gamma_1 \cup \Gamma_2 \vdash_{\mathcal{L}_{\mathbf{QNP}}} (\alpha_1 \to \alpha_2) \to (\beta_1 \to \beta_2) \tag{5.1}$$

$$\Gamma_1 \cup \Gamma_2 \vdash_{\mathcal{L}_{\mathbf{QNP}}} (\beta_1 \to \beta_2) \to (\alpha_1 \to \alpha_2) \tag{5.2}$$

$$\Gamma_1 \cup \Gamma_2 \vdash_{\mathcal{L}_{\mathbf{QNP}}} \sim(\alpha_1 \to \alpha_2) \to \sim(\beta_1 \to \beta_2) \tag{5.3}$$

$$\Gamma_1 \cup \Gamma_2 \vdash_{\mathcal{L}_{\mathbf{QNP}}} \sim(\beta_1 \to \beta_2) \to \sim(\alpha_1 \to \alpha_2) \tag{5.4}$$

$$\Gamma_1 \cup \Gamma_2 \vdash_{\mathcal{L}_{\mathbf{QNP}}} (\alpha_1 * \alpha_2) \to (\beta_1 * \beta_2) \tag{5.5}$$

$$\Gamma_1 \cup \Gamma_2 \vdash_{\mathcal{L}_{\mathbf{QNP}}} (\beta_1 * \beta_2) \to (\alpha_1 * \alpha_2) \tag{5.6}$$

$$\Gamma_1 \cup \Gamma_2 \vdash_{\mathcal{L}_{\mathbf{QNP}}} \sim(\alpha_1 * \alpha_2) \to \sim(\beta_1 * \beta_2) \tag{5.7}$$

$$\Gamma_1 \cup \Gamma_2 \vdash_{\mathcal{L}_{\mathbf{QNP}}} \sim(\beta_1 * \beta_2) \to \sim(\alpha_1 * \alpha_2) \tag{5.8}$$

The item (5.2), follows the same line of reasoning from (5.1), so we will only show item (5.1).

1. $\alpha_1 \to \alpha_2$	Hypothesis
2. β_1	Hypothesis
3. $\beta_1 \to \alpha_1$	Premise
4. α_1	MP, 2, 3
5. α_2	MP, 3, 5
6. $\alpha_2 \to \beta_2$	Premise
7. β_2	MP, 5, 6
8. $\beta_1 \to \beta_2$	DT, 2–7
9. $(\alpha_1 \to \alpha_2) \to (\beta_1 \to \beta_2)$	DT, 1–8

The item (5.4), follows the same line of reasoning from (5.3), so we will only show item (5.3).

1. $\sim(\alpha_1 \to \alpha_2)$	Hypothesis
2. $\sim(\alpha_1 \to \alpha_2) \to \sim\alpha_2$	Ax8
3. $\sim\alpha_2$	MP, 1, 2
4. $\sim\alpha_2 \to \sim\beta_2$	Premise
5. $\sim\beta_2$	MP, 3, 4
6. $\sim(\alpha_1 \to \alpha_2) \to \sim\sim\alpha_1$	Ax9
7. $\sim\sim\alpha_1$	MP, 1, 6
8. $(\alpha_1 \to \beta_1) \to (\sim\sim\alpha_1 \to \sim\sim\beta_1)$	Ax4
9. $\alpha_1 \to \beta_1$	Premise
10. $\sim\sim\alpha_1 \to \sim\sim\beta_1$	MP, 8, 9
11. $\sim\sim\beta_1$	MP, 7, 10
12. $\sim\sim\beta_1 \to (\sim\beta_2 \to \sim(\beta_1 \to \beta_2))$	Ax7
13. $\sim\beta_2 \to \sim(\beta_1 \to \beta_2)$	MP, 11, 12
14. $\sim(\beta_1 \to \beta_2)$	MP, 5, 13
15. $\sim(\alpha_1 \to \alpha_2) \to \sim(\beta_1 \to \beta_2)$	DT, 1–14

The item (5.6), follows the same line of reasoning from (5.5), so we will only show item (5.5).

1. $\alpha_1 * \alpha_2$	Hypothesis
2. $(\alpha_1 * \alpha_2) \to \alpha_1$	Ax11
3. α_1	MP, 1, 2
4. $\alpha_1 \to \beta_1$	Premise
5. β_1	MP, 3, 4
6. $(\alpha_1 * \alpha_2) \to \alpha_2$	Ax12
7. α_2	MP, 1, 6

8. $\alpha_2 \to \beta_2$ Premise
9. β_2 MP, 7, 8
10. $\beta_1 \to (\beta_2 \to (\beta_1 * \beta_2))$ Ax13
11. $\beta_2 \to (\beta_1 * \beta_2)$ MP, 5, 10
12. $\beta_1 * \beta_2$ MP, 9, 11
13. $(\alpha_1 * \alpha_2) \to (\beta_1 * \beta_2)$ DT, 1–12

The item (5.8), follows the same line of reasoning from (5.7), so we will only show item (5.7).

1. $\sim(\alpha_1 * \alpha_2)$	Hypothesis
2. $\sim(\alpha_1 * \alpha_2) \to ((\alpha_1 \to \sim\alpha_2) * (\alpha_2 \to \sim\alpha_1))$	Ax14 ($\to$)
3. $(\alpha_1 \to \sim\alpha_2) * (\alpha_2 \to \sim\alpha_1)$	MP, 1, 2
4. $((\alpha_1 \to \sim\alpha_2) * (\alpha_2 \to \sim\alpha_1)) \to (\alpha_1 \to \sim\alpha_2)$	Ax11
5. $\alpha_1 \to \sim\alpha_2$	MP, 3, 4
6. $\sim\alpha_2 \to \sim\beta_2$	Premise
7. $\alpha_1 \to \sim\beta_2$	Lemma 4, 5, 6
8. $((\alpha_1 \to \sim\alpha_2) * (\alpha_2 \to \sim\alpha_1)) \to (\alpha_2 \to \sim\alpha_1)$	Ax12
9. $\alpha_2 \to \sim\alpha_1$	MP, 3, 8
10. $\sim\alpha_1 \to \sim\beta_1$	Premise
11. $\alpha_2 \to \sim\beta_1$	Lemma 4, 9, 10
12. β_1	Hypothesis
13. $\beta_1 \to \alpha_1$	Premise
14. α_1	MP, 12, 13
15. $\sim\beta_2$	MP, 7, 14
16. $\beta_1 \to \sim\beta_2$	DT, 12–15
17. β_2	Hypothesis
18. $\beta_2 \to \alpha_2$	Premise
19. α_2	MP, 17, 18
20. $\sim\beta_1$	MP, 11, 19
21. $\beta_2 \to \sim\beta_1$	DT, 17–20
22. $(\beta_1 \to \sim\beta_2) \to ((\beta_2 \to \sim\beta_1) \to ((\beta_1 \to \sim\beta_2) * (\beta_2 \to \sim\beta_1)))$	Ax13
23. $(\beta_2 \to \sim\beta_1) \to ((\beta_1 \to \sim\beta_2) * (\beta_2 \to \sim\beta_1))$	MP, 16, 22
24. $(\beta_1 \to \sim\beta_2) * (\beta_2 \to \sim\beta_1)$	MP, 21, 23
25. $((\beta_1 \to \sim\beta_2) * (\beta_2 \to \sim\beta_1)) \to \sim(\beta_1 * \beta_2)$	Ax14 ($\leftarrow$)
26. $\sim(\beta_1 * \beta_2)$	MP, 24, 25
27. $\sim(\alpha_1 * \alpha_2) \to \sim(\beta_1 * \beta_2)$	DT, 1–26

□

Having proved that our calculus is algebraizable in the sense Blok and Pigozzi, we have (see [4]), a corresponding equivalent algebraic semantics $\mathtt{Alg}^*(\mathcal{L}_{\mathbf{QNP}})$ which satisfies the following equations and quasi-equations:

1. $E(p)$ for each $p \in \mathbf{Ax}$.
2. $E(\Delta(p, p))$.
3. $E(p)$ and $E(p \to q)$ implies $E(q)$.
4. $E(\Delta(p, q))$ implies $p = q$.

As an example of the notation $E(p)$ above, for each axiom $p \in \mathbf{Ax}$, the class of algebras $\mathtt{Alg}^*(\mathcal{L}_{\mathbf{QNP}})$ must satisfy $p = p \to p$. Taking **Ax3** as an example, the class $\mathtt{Alg}^*(\mathcal{L}_{\mathbf{QNP}})$ has $x \to \sim\sim x = (x \to \sim\sim x) \to (x \to \sim\sim x)$ as one of its equations.

Now, in order to prove that the class of algebras $\mathtt{Alg}^*(\mathcal{L}_{\mathbf{QNP}})$ is term-equivalent to the class of **QNP** (Definition 51), that is, the content of the next propositions.

Proposition 10. $\mathtt{Alg}^*(\mathcal{L}_{\mathbf{QNP}}) \subseteq \mathcal{V}_{\mathbf{QNP}}$.

Proof. It is easy to see that **QNPa** is true for $\mathbf{A} \in \mathtt{Alg}^*(\mathcal{L}_{\mathbf{QNP}})$.

To prove **QNPb**, we need to show that the commutative, associative laws and identity element laws holds for every $\mathbf{A} \in \mathtt{Alg}^*(\mathcal{L}_{\mathbf{QNP}})$, as well as $x^2 = x^3$.

- Commutative law: $x * y = y * x$.
    1. $(\alpha * \beta) \to (\beta * \alpha)$

1. $\alpha * \beta$	Hypothesis
2. $(\alpha * \beta) \to \alpha$	Ax11
3. α	MP, 1, 2
4. $(\alpha * \beta) \to \beta$	Ax12
5. β	MP, 1, 4
6. $\beta \to (\alpha \to (\beta * \alpha))$	Ax13
7. $\alpha \to (\beta * \alpha)$	MP, 5, 6
8. $\beta * \alpha$	MP, 3, 7
9. $(\alpha * \beta) \to (\beta * \alpha)$	DT, 1–8

    2. $(\beta * \alpha) \to (\alpha * \beta)$, this is an instantiation of previous item.

3. $\sim(\alpha * \beta) \to \sim(\beta * \alpha)$

1.	$\sim(\alpha * \beta)$	Hypothesis
2.	$\sim(\alpha * \beta) \to ((\alpha \to \sim\beta) * (\beta \to \sim\alpha))$	Ax14 ($\to$)
3.	$(\alpha \to \sim\beta) * (\beta \to \sim\alpha)$	MP, 1, 2
4.	$((\alpha \to \sim\beta) * (\beta \to \sim\alpha)) \to (\alpha \to \sim\beta)$	Ax11
5.	$\alpha \to \sim\beta$	MP, 3, 4
6.	$((\alpha \to \sim\beta) * (\beta \to \sim\alpha)) \to (\beta \to \sim\alpha)$	Ax12
7.	$\beta \to \sim\alpha$	MP, 3, 6
8.	$(\beta \to \sim\alpha) \to ((\alpha \to \sim\beta) \to ((\beta \to \sim\alpha) * (\alpha \to \sim\beta)))$	Ax13
9.	$(\alpha \to \sim\beta) \to ((\beta \to \sim\alpha) * (\alpha \to \sim\beta))$	MP, 7, 8
10.	$(\beta \to \sim\alpha) * (\alpha \to \sim\beta)$	MP, 5, 9
11.	$((\beta \to \sim\alpha) * (\alpha \to \sim\beta)) \to \sim(\beta * \alpha)$	Ax14 ($\leftarrow$)
12.	$\sim(\beta * \alpha)$	MP 10, 11
13.	$\sim(\alpha * \beta) \to \sim(\beta * \alpha)$	DT, 1–12

4. $\sim(\beta * \alpha) \to \sim(\alpha * \beta)$, this is an instantiation of previous item.

- Associative law: $x * (y * z) = (x * y) * z$.

1. $(\alpha * (\beta * \gamma)) \to ((\alpha * \beta) * \gamma)$

1.	$\alpha * (\beta * \gamma)$	Hypothesis
2.	$(\alpha * (\beta * \gamma)) \to \alpha$	Ax11
3.	α	MP, 1, 2
4.	$(\alpha * (\beta * \gamma)) \to (\beta * \gamma)$	Ax12
5.	$(\beta * \gamma) \to \beta$	Ax11
6.	$(\alpha * (\beta * \gamma)) \to \beta$	Lemma 4, 4, 5
7.	β	MP, 1, 6
8.	$\alpha \to (\beta \to (\alpha * \beta))$	Ax13
9.	$\beta \to (\alpha * \beta)$	MP, 3, 8
10.	$\alpha * \beta$	MP, 7, 9
11.	$(\beta * \gamma) \to \gamma$	Ax12
12.	$(\alpha * (\beta * \gamma)) \to \gamma$	Lemma 4, 4, 11
13.	γ	MP, 1, 12
14.	$(\alpha * \beta) \to (\gamma \to ((\alpha * \beta) * \gamma))$	Ax13
15.	$\gamma \to ((\alpha * \beta) * \gamma)$	MP, 10, 14
16.	$(\alpha * \beta) * \gamma$	MP, 13, 15
17.	$(\alpha * (\beta * \gamma)) \to ((\alpha * \beta) * \gamma)$	DT, 1–16

2. $((\alpha * \beta) * \gamma) \to (\alpha * (\beta * \gamma))$

1. $(\alpha * \beta) * \gamma$	Hypothesis
2. $((\alpha * \beta) * \gamma) \to (\alpha * \beta)$	Ax11
3. $(\alpha * \beta) \to \alpha$	Ax11
4. $((\alpha * \beta) * \gamma) \to \alpha$	Lemma 4, 2, 3
5. α	MP, 1, 4
6. $(\alpha * \beta) \to \beta$	Ax12
7. $((\alpha * \beta) * \gamma) \to \beta$	Lemma 4, 2, 6
8. β	MP, 1, 7
9. $((\alpha * \beta) * \gamma) \to \gamma$	Ax12
10. γ	MP, 1, 9
11. $\beta \to (\gamma \to (\beta * \gamma))$	Ax13
12. $\gamma \to (\beta * \gamma)$	MP, 8, 11
13. $\beta * \gamma$	MP, 10, 12
14. $\alpha \to ((\beta * \gamma) \to (\alpha * (\beta * \gamma)))$	Ax13
15. $(\beta * \gamma) \to (\alpha * (\beta * \gamma))$	MP, 5, 14
16. $\alpha * (\beta * \gamma)$	MP, 13, 15
17. $((\alpha * \beta) * \gamma) \to (\alpha * (\beta * \gamma))$	DT, 1–16

3. $\sim(\alpha * (\beta * \gamma)) \to \sim((\alpha * \beta) * \gamma)$ this is **Ax16** ($\to$).

4. $\sim((\alpha * \beta) * \gamma) \to \sim(\alpha * (\beta * \gamma))$ this is **Ax16** ($\leftarrow$).

- Identity element: $x * 1 = x$.

1. $(\alpha * \top) \to \alpha$ this is **Ax11**.

2. $\alpha \to (\alpha * \top)$

1. α	Hypothesis
2. $\alpha \to (\top \to (\alpha * \top))$	Ax13
3. $\top \to (\alpha * \top)$	MP, 1, 3
4. $\alpha \to \alpha$	Proposition 1
5. $\top$	Definition, 4
6. $\alpha * \top$	MP, 3, 5
7. $\alpha \to (\alpha * \top)$	DT, 1–6

3. $\sim(\alpha * \top) \to \sim\alpha$

1. $\sim(\alpha * \top)$	Hypothesis
2. $\sim(\alpha * \top) \to ((\alpha \to \sim\top) * (\top \to \sim\alpha))$	Ax14 ($\to$)
3. $(\alpha \to \sim\top) * (\top \to \sim\alpha)$	MP, 1, 2

4. $((\alpha \to \sim\top) * (\top \to \sim\alpha)) \to (\top \to \sim\alpha)$	Ax12
5. $\top \to \sim\alpha$	MP, 3, 4
6. $\alpha \to \alpha$	Proposition 1
7. $\top$	Definition, 6
8. $\sim\alpha$	MP, 5, 7
9. $\sim(\alpha * \top) \to \sim\alpha$	DT, 1–8

4. $\sim\alpha \to \sim(\alpha * \top)$

1. $\sim\alpha$	Hypothesis
2. $((\alpha \to \sim\top) * (\top \to \sim\alpha)) \to \sim(\alpha * \top)$	Ax14 ($\to$)
3. α	Hypothesis
4. $\alpha \to \sim\sim\alpha$	Ax5
5. $\sim\sim\alpha \to (\sim\alpha \to \sim(\alpha \to \alpha))$	Ax7
6. $\alpha \to (\sim\alpha \to \sim(\alpha \to \alpha))$	Lemma 4, 4, 5
7. $\sim\alpha \to \sim(\alpha \to \alpha)$	MP, 3, 6
8. $\sim(\alpha \to \alpha)$	MP, 1, 7
9. $\sim\top$	Definition, 8
10. $\alpha \to \sim\top$	DT, 3–9
11. $\top$	Hypothesis
12. $\sim\alpha$	Repetition, 1
13. $\top \to \sim\alpha$	DT, 11–12
14. $(\alpha \to \sim\top) \to ((\top \to \sim\alpha) \to ((\alpha \to \sim\top) * (\top \to \sim\alpha)))$	Ax13
15. $(\top \to \sim\alpha) \to ((\alpha \to \sim\top) * (\top \to \sim\alpha))$	MP, 10, 14
16. $(\alpha \to \sim\top) * (\top \to \sim\alpha)$	MP, 13, 15
17. $\sim(\alpha * \top)$	MP, 2, 16
18. $\sim\alpha \to \sim(\alpha * \top)$	DT, 1–17

Also, we need to prove that $x^2 = x^3$.

1. $(\alpha * \alpha * \alpha) \to (\alpha * \alpha)$, that is, $((\alpha * \alpha) * \alpha) \to (\alpha * \alpha)$ this is **Ax11**.

2. $(\alpha * \alpha) \to (\alpha * \alpha * \alpha)$

1. $\alpha * \alpha$	Hypothesis
2. $(\alpha * \alpha) \to (\alpha \to (\alpha * \alpha * \alpha))$	Ax13
3. $\alpha \to (\alpha * \alpha * \alpha)$	MP, 1, 2
4. $(\alpha * \alpha) \to \alpha$	Ax11
5. α	MP, 1, 4
6. $\alpha * \alpha * \alpha$	MP, 3, 5
7. $(\alpha * \alpha) \to (\alpha * \alpha * \alpha)$	DT, 1–6

3. $\sim(\alpha * \alpha * \alpha) \to \sim(\alpha * \alpha)$

1.	$\sim(\alpha * \alpha * \alpha)$	Hypothesis
2.	$((\alpha \to \sim\alpha) * (\alpha \to \sim\alpha)) \to \sim(\alpha * \alpha)$	Ax14 ($\leftarrow$)
3.	$\sim(\alpha * \alpha * \alpha) \to (((\alpha * \alpha) \to \sim\alpha) * (\alpha \to \sim(\alpha * \alpha)))$	Ax14 ($\to$)
4.	$((\alpha * \alpha) \to \sim\alpha) * (\alpha \to \sim(\alpha * \alpha))$	MP, 1, 3
5.	$((\alpha * \alpha) \to \sim\alpha) * (\alpha \to \sim(\alpha * \alpha)) \to ((\alpha * \alpha) \to \sim\alpha)$	Ax11
6.	$(\alpha * \alpha) \to \sim\alpha$	MP, 4, 5
7.	α	Hypothesis
8.	$\alpha \to (\alpha \to (\alpha * \alpha))$	Ax13
9.	$\alpha \to (\alpha * \alpha)$	MP, 7, 8
10.	$\alpha * \alpha$	MP, 7, 9
11.	$\sim\alpha$	MP, 6, 10
12.	$\alpha \to \sim\alpha$	DT, 7–11
13.	$(\alpha \to \sim\alpha) \to ((\alpha \to \sim\alpha) \to ((\alpha \to \sim\alpha) * (\alpha \to \sim\alpha)))$	Ax13
14.	$(\alpha \to \sim\alpha) \to ((\alpha \to \sim\alpha) * (\alpha \to \sim\alpha))$	MP, 12, 13
15.	$(\alpha \to \sim\alpha) * (\alpha \to \sim\alpha)$	MP, 12, 14
16.	$\sim(\alpha * \alpha)$	MP, 2, 15
17.	$\sim(\alpha * \alpha * \alpha) \to \sim(\alpha * \alpha)$	DT, 1–16

4. $\sim(\alpha * \alpha) \to \sim(\alpha * \alpha * \alpha)$

1.	$\sim(\alpha * \alpha)$	Hypothesis
2.	$(((\alpha * \alpha) \to \sim\alpha) * (\alpha \to \sim(\alpha * \alpha))) \to \sim(\alpha * \alpha * \alpha)$	Ax14 ($\leftarrow$)
3.	$\alpha * \alpha$	Hypothesis
4.	$(\alpha * \alpha) \to \alpha$	Ax11
5.	α	MP, 3, 4
6.	$\sim(\alpha * \alpha) \to ((\alpha \to \sim\alpha) * (\alpha \to \sim\alpha))$	Ax14 ($\to$)
7.	$(\alpha \to \sim\alpha) * (\alpha \to \sim\alpha)$	MP, 1, 6
8.	$((\alpha \to \sim\alpha) * (\alpha \to \sim\alpha)) \to (\alpha \to \sim\alpha)$	Ax11
9.	$\alpha \to \sim\alpha$	MP, 7, 8
10.	$\sim\alpha$	MP, 5, 9
11.	$(\alpha * \alpha) \to \sim\alpha$	DT, 3–10
12.	α	Hypothesis
13.	$\sim(\alpha * \alpha)$	Repetition, 1
14.	$\alpha \to \sim(\alpha * \alpha)$	DT, 12–13
15.	$((\alpha * \alpha) \to \sim\alpha) \to ((\alpha \to \sim(\alpha * \alpha)) \to (((\alpha * \alpha) \to \sim\alpha) * (\alpha \to \sim(\alpha * \alpha))))$	Ax13
16.	$(\alpha \to \sim(\alpha * \alpha)) \to (((\alpha * \alpha) \to \sim\alpha) * (\alpha \to \sim(\alpha * \alpha)))$	MP, 11, 15
17.	$((\alpha * \alpha) \to \sim\alpha) * (\alpha \to \sim(\alpha * \alpha))$	MP, 14, 16
18.	$\sim(\alpha * \alpha * \alpha)$	MP, 2, 17
19.	$\sim(\alpha * \alpha) \to \sim(\alpha * \alpha * \alpha)$	DT, 1–18

For **QNPc.1**, we need to prove that $(x * y) \to z = x \to (y \to z)$.

1. $((\alpha * \beta) \to \gamma) \to (\alpha \to (\beta \to \gamma))$

1. $(\alpha * \beta) \to \gamma$	Hypothesis
2. α	Hypothesis
3. β	Hypothesis
4. $\alpha \to (\beta \to (\alpha * \beta))$	Ax13
5. $\beta \to (\alpha * \beta)$	MP, 2, 4
6. $\alpha * \beta$	MP, 3, 5
7. γ	MP, 1, 6
8. $\beta \to \gamma$	DT, 3–7
9. $\alpha \to (\beta \to \gamma)$	DT, 2–8
10. $((\alpha * \beta) \to \gamma) \to (\alpha \to (\beta \to \gamma))$	DT, 1–9

2. $(\alpha \to (\beta \to \gamma)) \to ((\alpha * \beta) \to \gamma)$

1. $\alpha \to (\beta \to \gamma)$	Hypothesis
2. $\alpha * \beta$	Hypothesis
3. $(\alpha * \beta) \to \alpha$	Ax11
4. $(\alpha * \beta) \to (\beta \to \gamma)$	Lemma 4, 1, 3
5. $\beta \to \gamma$	MP, 2, 4
6. $(\alpha * \beta) \to \beta$	Ax12
7. β	MP, 2, 6
8. γ	MP, 5, 7
9. $(\alpha * \beta) \to \gamma$	DT, 2–8
10. $(\alpha \to (\beta \to \gamma)) \to ((\alpha * \beta) \to \gamma)$	DT, 1–9

3. $\sim((\alpha * \beta) \to \gamma) \to \sim(\alpha \to (\beta \to \gamma))$ this is the **Ax17** ($\to$).

4. $\sim(\alpha \to (\beta \to \gamma)) \to \sim((\alpha * \beta) \to \gamma)$ this is the **Ax17** ($\leftarrow$).

For **QNPc.2**, we need to prove that $x \to (y * z) \equiv (x \to y) * (x \to z)$.

1. $(\alpha \to (\beta * \gamma)) \to ((\alpha \to \beta) * (\alpha \to \gamma))$ this is the **Ax15** ($\to$).

2. $((\alpha \to \beta) * (\alpha \to \gamma)) \to (\alpha \to (\beta * \gamma))$ this is the **Ax15** ($\leftarrow$).

For **QNPc.3**, we need to prove that $\sim(x * y) \equiv (x \to \sim y) * (y \to \sim x)$.

1. $\sim(\alpha * \beta) \to ((\alpha \to \sim\beta) * (\beta \to \sim\alpha))$ this is the **Ax14** ($\to$).

2. $((\alpha \to \sim\beta) * (\beta \to \sim\alpha)) \to \sim(\alpha * \beta)$ this is the **Ax14** ($\leftarrow$).

For **QNPc.4**, we need to prove that $\sim(x \to y) \equiv \sim\sim x * \sim y$.

1. $\sim(\alpha \to \beta) \to (\sim\sim\alpha * \sim\beta)$

1. $\sim(\alpha \to \beta)$	Hypothesis
2. $\sim(\alpha \to \beta) \to \sim\sim\alpha$	Ax13
3. $\sim\sim\alpha$	MP, 1, 2
4. $\sim(\alpha \to \beta) \to \sim\beta$	Ax12
5. $\sim\beta$	MP, 1, 4
6. $\sim\sim\alpha \to (\sim\beta \to (\sim\sim\alpha * \sim\beta))$	Ax9
7. $\sim\beta \to (\sim\sim\alpha * \sim\beta)$	MP, 3, 6
8. $\sim\sim\alpha * \sim\beta$	MP, 5, 7
9. $\sim(\alpha \to \beta) \to (\sim\sim\alpha * \sim\beta)$	DT, 1–8

2. $(\sim\sim\alpha * \sim\beta) \to \sim(\alpha \to \beta)$

1. $\sim\sim\alpha * \sim\beta$	Hypothesis
2. $(\sim\sim\alpha * \sim\beta) \to \sim\sim\alpha$	Ax4
3. $\sim\sim\alpha$	MP, 1, 2
4. $(\sim\sim\alpha * \sim\beta) \to \sim\beta$	Ax11
5. $\sim\beta$	MP, 1, 4
6. $\sim\sim\alpha \to (\sim\beta \to \sim(\alpha \to \beta))$	Ax7
7. $\sim\beta \to \sim(\alpha \to \beta)$	MP, 3, 6
8. $\sim(\alpha \to \beta)$	MP, 5, 7
9. $(\sim\sim\alpha * \sim\beta) \to \sim(\alpha \to \beta)$	DT, 1–8

□

Proposition 11. $\mathcal{V}_{\mathbf{QNP}} \subseteq \mathtt{Alg}^*(\mathcal{L}_{\mathbf{QNP}})$.

Proof. Let $\mathbf{A} \in \mathbf{QNP}$, and let $a, b, c \in A$ be generic elements. By Theorem 12, we assume that $\mathbf{A}$ is a twist-structure, and from now on we also denote $a = \langle a_1, a_2 \rangle$, $b = \langle b_1, b_2 \rangle$ and $c = \langle c_1, c_2 \rangle$. In the case of $E(a \to b)$ saying this is equivalent to proving that $\pi_1(a) \leq \pi_1(b)$, this is, $a_1 \leq b_1$.

The axioms **Ax1-Ax5** are also present in $\mathcal{L}_{\mathbf{QN4}}$, so their checks will be omitted. For equations and quasi-equations of $\mathtt{Alg}^*(\mathcal{L}_{\mathbf{QNP}})$, we have

- $E((a \odot (a \to b)) \to (a \odot b))$

  On the one hand, $\pi_1[a \odot (a \to b)] = \pi_1[\sim(a \to \sim(a \to b))] = \pi_1[\sim(\langle a_1, a_2\rangle \to \sim(\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle))] = \pi_1[\sim(\langle a_1, a_2\rangle \to \sim\langle a_1 \to b_1, \Box a_1 \wedge b_2\rangle)] = \pi_1[\sim(\langle a_1, a_2\rangle \to \langle \Box a_1 \wedge b_2, \Box(a_1 \to b_1)\rangle)] = \pi_1[\sim\langle a_1 \to (\Box a_1 \wedge a_2), \Box a_1 \wedge \Box(a_1 \to b_1)\rangle] = \pi_1[\langle \Box a_1 \wedge \Box(a_1 \to b_1), \Box(a_1 \to (\Box a_1 \wedge a_2))\rangle] = \Box a_1 \wedge \Box(a_1 \to b_1) = \Box(a_1 \wedge (a_1 \to b_1)) = \Box(a_1 \wedge b_1)$.

  On the other hand, $\pi_1[a \odot b] = \pi_1[\sim(a \to \sim b)] = \pi_1[\sim(\langle a_1, a_2\rangle \to \sim\langle b_1, b_2\rangle)] = \pi_1[\sim(\langle a_1, a_2\rangle \to \langle b_2, \Box b_1\rangle)] = \pi_1[\sim\langle a_1 \to b_2, \Box a_1 \wedge \Box b_1\rangle] = \pi_1[\langle \Box a_1 \wedge \Box b_1, \Box(a_1 \to b_2)\rangle] = \Box a_1 \wedge \Box b_1 = \Box(a_1 \wedge b_1)$.

- $E(\sim\sim a \to (\sim b \to \sim(a \to b)))$

  On the one hand, $\pi_1[\sim\sim a] = \pi_1[\sim\sim\langle a_1, a_2\rangle] = \pi_1[\sim\langle a_2, \Box a_1\rangle] = \pi_1[\Box a_1, \Box a_2] = \Box a_1$.

  On the other hand, $\pi_1[\sim b \to \sim(a \to b)] = \pi_1[\sim\langle b_1, b_2\rangle \to \sim(\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle)] = \pi_1[\langle b_2, \Box b_1\rangle \to \sim\langle a_1 \to b_1, \Box a_1 \wedge b_2\rangle] = \pi_1[\langle b_2, \Box b_1\rangle \to \langle \Box a_1 \wedge b_2, \Box(a_1 \to b_1)\rangle] = \pi_1[\langle b_2 \to (\Box a_1 \wedge b_2), \Box b_2 \wedge \Box(a_1 \to b_1)\rangle] = b_2 \to (\Box a_1 \wedge b_2)$.

- $E(\sim(a \to b) \to \sim b)$

  On the one hand, $\pi_1[\sim(a \to b)] = \pi_1[\sim(\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle)] = \pi_1[\sim\langle a_1 \to b_1, \Box a_1 \wedge b_2\rangle] = \pi_1[\Box a_1 \wedge b_2, \Box(a_1 \to b_1)] = \Box a_1 \wedge b_2$.

  On the other hand, $\pi_1[\sim b] = \pi_1[\sim\langle b_1, b_2\rangle] = \pi_1[\langle b_2, \Box b_1\rangle] = b_2$.

- $E(\sim(a \to b) \to \sim\sim a)$

  On the one hand, $\pi_1[\sim(a \to b)] = \pi_1[\sim(\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle)] = \pi_1[\sim\langle a_1 \to b_1, \Box a_1 \wedge b_2\rangle] = \pi_1[\Box a_1 \wedge b_2, \Box(a_1 \to b_1)] = \Box a_1 \wedge b_2$.

  On the other hand, $\pi_1[\sim\sim a] = \pi_1[\sim\sim\langle a_1, a_2\rangle] = \pi_1[\sim\langle a_2, \Box a_1\rangle] = \pi_1[\Box a_1, \Box a_2] = \Box a_1$.

- $E(\sim(a \to a) \to b)$

  On the one hand, $\pi_1[\sim(a \to a)] = \pi_1[\sim(\langle a_1, a_2\rangle \to \langle a_1, a_2\rangle)] = \pi_1[\sim\langle a_1 \to a_1, \Box a_1 \wedge$

$a_2\rangle] = \pi_1[\langle \Box a_1 \wedge a_2, \Box(a_1 \to a_1)\rangle] = \Box a_1 \wedge a_2 = \Box a_1 \wedge \Box a_2 = \Box(a_1 \wedge a_2) = \Box 0 = 0.$

On the other hand, $\pi_1[b] = \pi_1[\langle b_1, b_2\rangle] = b_1$.

- $E((a * b) \to a)$

  On the one hand, $\pi_1[a * b] = \pi_1[\langle a_1, a_2\rangle * \langle b_1, b_2\rangle] = \pi_1[\langle a_1 \wedge b_1, (a_1 \to b_2) \wedge (b_1 \to a_2)\rangle] = a_1 \wedge b_1$.

  On the other hand, $\pi_1[a] = \pi_1[\langle a_1, a_2\rangle] = a_1$.

- $E((a * b) \to b)$

  On the one hand, $\pi_1[a * b] = \pi_1[\langle a_1, a_2\rangle * \langle b_1, b_2\rangle] = \pi_1[\langle a_1 \wedge b_1, (a_1 \to b_2) \wedge (b_1 \to a_2)\rangle] = a_1 \wedge b_1$.

  On the other hand, $\pi_1[b] = \pi_1[\langle b_1, b_2\rangle] = b_1$.

- $E(a \to (b \to (a * b)))$

  On the one hand, $\pi_1[a] = \pi_1[\langle a_1, a_2\rangle] = a_1$.

  On the other hand, $\pi_1[b \to (a * b)] = \pi_1[\langle b_1, b_2\rangle \to (\langle a_1, a_2\rangle * \langle b_1, b_2\rangle)] = \pi_1[\langle b_1, b_2\rangle \to \langle a_1 \wedge b_1, (a_1 \to b_2) \wedge (b_1 \to a_2)\rangle] = \pi_1[\langle b_1 \to (a_1 \wedge b_1), \Box b_1 \wedge ((a_1 \to b_2) \wedge (b_1 \to a_2))\rangle] = b_1 \to (a_1 \wedge b_1)$.

  In the case of $E(a \leftrightarrow b)$ saying this is equivalent to proving that $\pi_1(a) = \pi_1(b)$, this is, $a_1 = b_1$. So,

- $E(\sim(a * b) \leftrightarrow ((a \to \sim b) * (b \to \sim a)))$

  On the one hand, $\pi_1[\sim(a * b)] = \pi_1[\sim(\langle a_1, a_2\rangle * \langle b_1, b_2\rangle)] = \pi_1[\sim\langle a_1 \wedge b_1, (a_1 \to b_2) \wedge (b_1 \to a_2)\rangle] = \pi_1[\langle (a_1 \to b_2) \wedge (b_1 \to a_2), \Box(a_1 \wedge b_1)\rangle] = (a_1 \to b_2) \wedge (b_1 \to a_2)$.

  On the other hand, $\pi_1[(a \to \sim b) * (b \to \sim a)] = \pi_1[(\langle a_1, a_2\rangle \to \sim\langle b_1, b_2\rangle) * (\langle b_1, b_2\rangle \to \sim\langle a_1, a_2\rangle)] = \pi_1[(\langle a_1, a_2\rangle \to \langle b_2, \Box b_1\rangle) * (\langle b_1, b_2\rangle \to \langle a_2, \Box a_1\rangle)] = \pi_1[\langle a_1 \to b_2, \Box a_1 \wedge \Box b_1\rangle * \langle b_1 \to a_2, \Box b_1 \wedge \Box a_1\rangle] = \pi_1[\langle (a_1 \to b_2) \wedge (b_1 \to a_2), ((a_1 \to b_2) \to (\Box b_1 \wedge \Box a_1)) \wedge ((b_1 \to a_2) \to (\Box a_1 \wedge \Box b_1))\rangle] = (a_1 \to b_2) \wedge (b_1 \to a_2)$.

- $E((a \to (b * c)) \leftrightarrow ((a \to b) * (a \to c)))$

  On the one hand, $\pi_1[(a \to (b * c)] = \pi_1[\langle a_1, a_2\rangle \to (\langle b_1, b_2\rangle * \langle c_1, c_2\rangle)] = \pi_1[\langle a_1, a_2\rangle \to$

$\langle b_1 \wedge c_1, (b_1 \to c_2) \to (c_1 \to b_2)\rangle] = \pi_1[\langle a_1 \to (b_1 \wedge c_1), \Box a_1 \wedge ((b_1 \to c_2) \to (c_1 \to b_2)))\rangle] = a_1 \to (b_1 \wedge c_1)$.

On the other hand, $\pi_1[(a \to b) * (a \to c)] = \pi_1[(\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle) * (\langle a_1, a_2\rangle \to \langle c_1, c_2\rangle)] = \pi_1[\langle a_1 \to b_1, \Box a_1 \wedge b_2\rangle * \langle a_1 \to c_1, \Box a_1 \wedge c_2\rangle] = \pi_1[(a_1 \to b_1) \wedge (a_1 \to c_1), ((a_1 \to b_1) \to (\Box a_1 \wedge c_2)) \to ((a_1 \to c_1) \to (\Box a_1 \wedge b_2))] = (a_1 \to b_1) \wedge (a_1 \to c_1) = a_1 \to (b_1 \wedge c_1)$.

- $E({\sim}(a * (b * c)) \leftrightarrow {\sim}((a * b) * c)))$

  On the one hand, $\pi_1[{\sim}(a*(b*c))] = \pi_1[{\sim}(\langle a_1, a_2\rangle*(\langle b_1, b_2\rangle*\langle c_1, c_2\rangle))] = \pi_1[{\sim}(\langle a_1, a_2\rangle* \langle b_1 \wedge c_1, (b_1 \to c_2) \wedge (c_1 \to b_2)\rangle)] = \pi_1[{\sim}\langle a_1 \wedge (b_1 \wedge c_1), (a_1 \to ((b_1 \to c_2) \wedge (c_1 \to b_2))) \wedge ((b_1 \wedge c_1) \to a_2)\rangle] = \pi_1[\langle (a_1 \to ((b_1 \to c_2) \wedge (c_1 \to b_2))) \wedge ((b_1 \wedge c_1) \to a_2), \Box(a_1 \wedge (b_1 \wedge c_1))\rangle] = (a_1 \to ((b_1 \to c_2) \wedge (c_1 \to b_2))) \wedge ((b_1 \wedge c_1) \to a_2)$.

  On the other hand, $\pi_1[{\sim}((a*b)*c)] = \pi_1[{\sim}((\langle a_1, a_2\rangle*\langle b_1, b_2\rangle)*\langle c_1, c_2\rangle)] = \pi_1[{\sim}(\langle a_1 \wedge b_1, (a_1 \to b_2) \wedge (b_1 \to a_2)\rangle*\langle c_1, c_2\rangle)] = \pi_1[{\sim}\langle (a_1 \wedge b_1) \wedge c_1, ((a_1 \wedge b_1) \to c_2) \wedge (c_1 \to ((a_1 \to b_2) \wedge (b_1 \to a_2)))\rangle] = \pi_1[\langle ((a_1 \wedge b_1) \to c_2) \wedge (c_1 \to ((a_1 \to b_2) \wedge (b_1 \to a_2))), \Box((a_1 \wedge b_1) \wedge c_1)\rangle] = ((a_1 \wedge b_1) \to c_2) \wedge (c_1 \to ((a_1 \to b_2) \wedge (b_1 \to a_2)))$.

We have to prove that $E(a \to a)$ and $E({\sim}\, a \to {\sim}\, a)$. These are easy to check.

We have to prove that if $E(a)$ and $E(a \to b)$ then $E(b)$. Therefore, we will use the fact that $|a| \to b = b$. Note that,

$$\begin{cases} |a| = |\langle a_1, a_2\rangle| = \langle a_1, a_2\rangle \to \langle a_1, a_2\rangle = \langle a_1 \to a_1, \Box a_1 \wedge a_2\rangle = \langle 1, \Box a_1 \wedge a_2\rangle \\ |a| \to b = \langle 1, \Box a_1 \wedge a_2\rangle \to \langle b_1, b_2\rangle = \langle 1 \to b_1, \Box 1 \wedge b_2\rangle = \langle b_1, b_2\rangle = b \end{cases}$$

Thus $(a \to a) \to b = b$, but we have that $a \to a = a$ and therefore $a \to b = b$, but as $a \to b = (a \to b) \to (a \to b)$ and $a \to b = b$, then $b = b \to b$ and this is what we wanted to prove.

We have to prove that if $E(\Delta(a, b))$ then $a = b$. So, $E(a \to b)$, $E(b \to a)$, $E({\sim}\, a \to {\sim}\, b)$, $E({\sim}\, b \to {\sim}\, a)$, we give us $a_1 \leq b_1$, $b_1 \leq a_1$, $a_2 \leq b_2$, $b_2 \leq a_2$, respectively. Therefore $a = b$. $\Box$

Corollary 2. The class of $\mathcal{V}_{\mathbf{QNP}}$ and the class of $\mathtt{Alg}^*(\mathcal{L}_{\mathbf{QNP}})$-algebras coincide.

5.2 $\{\sim, *, \Rightarrow, \wedge\}$-fragment

We will begin our section with definitions and important results for the understanding of the study of the $\{\sim, *, \Rightarrow, \wedge\}$-fragment of the quasi-Nelson logic.

Semihoops were introduced in [8, Def. 3.6] and can be defined as an algebra $\mathbf{A} = \langle A;\ \wedge, *, \Rightarrow, 1\rangle$ of type $\langle 2, 2, 2, 0\rangle$ such that:

1. $\langle A;\ \wedge, 1\rangle$ is a semilattice with order $\leq$ and 1 as top element.
2. $\langle A, \leq;\ *, \Rightarrow, 1\rangle$ is a pocrim.

The preceding definition is slightly more informative than the original one, but easily seen to be equivalent to it. A *hoop* [8, Rem. 3.11] may be defined as a semihoop $\langle A;\ \wedge, *, \Rightarrow, 1\rangle$ that satisfies the divisibility equation:

1. $x \wedge y = x * (x \Rightarrow y)$, $\forall\, x, y \in A$.

For further background on hoops, see [5, 3].

Definition 54 ([23], Def. 5.1)**.** A **quasi-Nelson semihoop (QNS)** is an algebra $\mathbf{A} = \langle A;\ *, \rightarrow, \wedge, \sim, 0, 1\rangle$ of type $\langle 2, 2, 2, 1, 0, 0\rangle$ such that:

(QNSa) $\langle A;\ *, \rightarrow, \sim, 0, 1\rangle$ is a quasi-Nelson pocrim.

(QNSb) $\langle A;\ \wedge, 0, 1\rangle$ is a bounded semilattice whose partial order coincides with that of the pocrim reduct of $\mathbf{A}$.

(QNSc) For all x, $y \in A$, we have:

(QNSc.1) $\sim\sim\sim x = \sim x$.

(QNSc.2) $\sim\sim(x \wedge y) = \sim\sim x \wedge \sim\sim y$.

(QNSc.3) $\sim\sim x \wedge (y \oplus z) = (x \wedge y) \oplus (x \wedge z)$.

(QNSc.4) $x \oplus y \equiv x^2 \oplus y^2$.

The class of all quasi-Nelson semihoops will be denoted by **QNS**. It is easy to verify that every member of **QNS** is, indeed, a semihoop in the terminology of [8], though not necessarily a hoop.

We are now ready to introduce the class of twist-algebras that correspond to quasi-Nelson semihoops.

Definition 55 ([23], Def. 5.3)**.** An algebra $\mathbf{S} = \langle S;\ \wedge, \oplus, \to, 0, 1\rangle$ is a **$\oplus$-implicative semilattice** such that:

1. $\langle S;\ \wedge, \to, \Box, 0, 1\rangle$ is a bounded implicative semilattice with a nucleus given by $\Box x := x \oplus x$.
2. $\langle S;\ \oplus\rangle$ is a commutative semigroup.
3. The following equations are satisfied:
   a) $x \oplus 1 = 1$.
   b) $\Box x = x \oplus 0 = x \oplus (x \wedge y)$.
   c) $x \leq x \oplus y = \Box x \oplus \Box y$.
   d) $\Box x \wedge (y \oplus z) = (x \wedge y) \oplus (x \wedge z)$.

Definition 56 ([23], Def. 5.6)**.** Let $\mathbf{S} = \langle S;\ \wedge, \oplus, \to, 0, 1\rangle$ be a $\oplus$-implicative semilattice. Define the algebra $\mathbf{S}^{\bowtie} = \langle S^{\bowtie};\ \wedge, *, \to, \sim, 0, 1\rangle$ with universe $S^{\bowtie} := \{\langle a_1, a_2\rangle \in S \times S : a_2 = \Box a_2,\ a_1 \wedge a_2 = 0\}$ and operations given, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in S \times S$, by:

$$0 := \langle 0, 1\rangle$$

$$1 := \langle 1, 0\rangle$$

$$\sim\langle a_1, a_2\rangle := \langle a_2, \Box a_1\rangle$$

$$\langle a_1, a_2\rangle * \langle b_1, b_2\rangle = \langle a_1 \wedge b_1, (a_1 \to b_2) \wedge (b_1 \to a_2)\rangle$$

$$\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle := \langle a_1 \to b_1, \Box a_1 \wedge b_2\rangle$$

$$\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle = \langle a_1 \wedge b_1, a_2 \oplus b_2\rangle$$

A **QNS twist-algebra** over $\mathbf{S}$ is any subalgebra $\mathbf{A} \leq \mathbf{S}^{\bowtie}$ satisfying $\pi_1[A] = S$.

Theorem 14 ([23], Thm. 5.9)**.** Every $\mathbf{A} \in \mathbf{QNS}$ is isomorphic to a QNS twist-algebra over $\mathbf{A}^{\bowtie}$ through the map $\iota : A \to A_{\bowtie} \times A_{\bowtie}$ given by $\iota(a) := \langle [a], [\sim a] \rangle$ for all $a \in A$.

5.2.1 A Hilbert-style calculus

In this subsection we introduce a Hilbert-style calculus that determines a logic, henceforth denoted by $\mathcal{L}_{\mathbf{QNS}}$. Our aim is to show that $\mathcal{L}_{\mathbf{QNS}}$ is algebraizable, and that its equivalent algebraic semantics is precisely the variety $\mathcal{V}_{\mathbf{QNS}}$.

The Hilbert-system for $\mathcal{L}_{\mathbf{QNS}}$ consists of the following axiom schemes together with the single inference rule of *modus ponens* (MP): $\alpha, \alpha \to \beta \vdash \beta$.

Ax1 $\alpha \to (\beta \to \alpha)$

Ax2 $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

Ax3 $\sim\sim\sim\alpha \to \sim\alpha$

Ax4 $(\alpha \to \beta) \to (\sim\sim\alpha \to \sim\sim\beta)$

Ax5 $\alpha \to \sim\sim\alpha$

Ax6 $(\alpha \odot (\alpha \to \beta)) \to (\alpha \odot \beta)$

Ax7 $\sim\sim\alpha \to (\sim\beta \to \sim(\alpha \to \beta))$

Ax8 $\sim(\alpha \to \beta) \to \sim\beta$

Ax9 $\sim(\alpha \to \beta) \to \sim\sim\alpha$

Ax10 $\sim(\alpha \to \alpha) \to \beta$

Ax11 $(\alpha * \beta) \to \alpha$

Ax12 $(\alpha * \beta) \to \beta$

Ax13 $\alpha \to (\beta \to (\alpha * \beta))$

Ax14 $\sim(\alpha * \beta) \leftrightarrow ((\alpha \to \sim\beta) * (\beta \to \sim\alpha))$

Ax15 $(\alpha \to (\beta * \gamma)) \leftrightarrow ((\alpha \to \beta) * (\alpha \to \gamma))$

Ax16 $\sim(\alpha * (\beta * \gamma)) \leftrightarrow \sim((\alpha * \beta) * \gamma))$

Ax17 $\sim((\alpha * \beta) \rightarrow \gamma) \leftrightarrow \sim(\alpha \rightarrow (\beta \rightarrow \gamma))$

Ax18 $(\alpha \wedge \beta) \rightarrow \alpha$

Ax19 $(\alpha \wedge \beta) \rightarrow \beta$

Ax20 $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \gamma) \rightarrow (\alpha \rightarrow (\beta \wedge \gamma)))$

Ax21 $\sim\alpha \rightarrow \sim(\alpha \wedge \beta)$

Ax22 $\sim(\alpha \wedge \beta) \rightarrow \sim(\beta \wedge \alpha)$

Ax23 $(\sim\alpha \rightarrow \sim\beta) \rightarrow (\sim(\alpha \wedge \beta) \rightarrow \sim\beta)$

Ax24 $(\sim\alpha \rightarrow \sim\beta) \rightarrow ((\sim\gamma \rightarrow \sim\theta) \rightarrow (\sim(\alpha \wedge \gamma) \rightarrow \sim(\beta \wedge \theta)))$

Ax25 $\sim(\alpha \rightarrow \beta) \leftrightarrow \sim\sim(\alpha \wedge \sim\beta)$

Ax26 $\sim(\alpha \wedge (\beta \wedge \gamma)) \leftrightarrow \sim((\alpha \wedge \beta) \wedge \gamma)$

Ax27 $\sim\sim(\alpha \wedge \beta) \leftrightarrow (\sim\sim\alpha \wedge \sim\sim\beta)$

Ax28 $\sim\sim\alpha \wedge (\beta \oplus \gamma) \leftrightarrow (\alpha \wedge \beta) \oplus (\alpha \wedge \gamma)$

Ax29 $\alpha \oplus \beta \leftrightarrow \alpha^2 \oplus \beta^2$

Axioms **Ax1-Ax17** together with modus ponens constitute an axiomatization of the $\mathcal{L}_{\mathbf{QNP}}$. Furthemore, axioms **Ax18-Ax26** are choose between the axioms of $\mathcal{L}_{\mathbf{QN4}}$, thus we prove that our calculus is algebraizable and that its equivalent algebraic semantics is the class of **QNS** as defined in Definition 54.

Remark 14. The Deduction Theorem holds for $\mathcal{L}_{\mathbf{QNS}}$.

5.2.2 $\mathcal{L}_{\mathbf{QNS}}$ is BP-Algebraizable

In this subsection we prove that the calculus introduced in the previous subsection is algebraizable in sense of Blok and Pigozzi. Using this result, we will axiomatize the equivalent algebraic semantics of $\mathcal{L}_{\mathbf{QNS}}$ via the algorithm of Theorem 8 and show that is term-equivalent to the variety $\mathcal{V}_{\mathbf{QNS}}$.

Theorem 15. $\mathcal{L}_{\mathbf{QNS}}$ is BP-algebraizable with $E(\alpha) := \{\alpha = \alpha \to \alpha\}$ and $\Delta(\alpha, \beta) := \{\alpha \to \beta, \beta \to \alpha, \sim\alpha \to \sim\beta, \sim\beta \to \sim\alpha\}$.

Proof. By Theorem 7, we have prove (**Ref**), (**MP**), (**Alg**) and (**Cong**). The first three follow the same reasoning of Theorem 11. As to (**Cong**), we need to prove it for each connective $\lambda \in \{\to, *, \wedge, \sim\}$. For $(\sim)$ and $(\wedge)$, we have the same reasoning of Theorem 11. Furthermore, for $(*)$ and $(\to)$, we have the same reasoning of Theorem 13. □

Having proved that our calculus is algebraizable in the sense Blok and Pigozzi, we have a corresponding equivalent algebraic semantics $\texttt{Alg}^*(\mathcal{L}_{\mathbf{QNS}})$ which satisfies the following equations and quasi-equations:

1. $E(p)$ for each $p \in \mathbf{Ax}$.
2. $E(\Delta(p, p))$.
3. $E(p)$ and $E(p \to q)$ implies $E(q)$.
4. $E(\Delta(p, q))$ implies $p = q$.

Now, in order to prove that the class of algebras $\texttt{Alg}^*(\mathcal{L}_{\mathbf{QNS}})$ is term-equivalent to the class of **QNS** (Definition 54), that is, the content of the next propositions.

Proposition 12. $\texttt{Alg}^*(\mathcal{L}_{\mathbf{QNS}}) \subseteq \mathcal{V}_{\mathbf{QNS}}$.

Proof. It is easy to see that **QNSa** is true for $\mathbf{A} \in \texttt{Alg}^*(\mathcal{L}_{\mathbf{QNS}})$.

For proving **QNSb**, we need to show that $\langle A;\ \wedge\rangle$ is a bounded semilattice.

- $\langle A;\ \wedge\rangle$ is a semilattice.
  1. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, see proof in Proposition 8.
  2. $x \wedge y = y \wedge x$, see proof in Proposition 8.
  3. $x \wedge x = x$, see proof in Proposition 8.
- $x \wedge 0 = 0$.
  1. $(x \wedge \bot) \to \bot$ this is **Ax19**.

2. $\bot \to (x \wedge \bot)$

1. $\bot$	Hypothesis
2. $\sim(\alpha \to \alpha) \to (\alpha \wedge \bot)$	Ax10
3. $\bot \to (\alpha \wedge \bot)$	Definition, 2
4. $\alpha \wedge \bot$	MP, 1, 3
5. $\bot \to (\alpha \wedge \bot)$	DT, 1–4

3. $\sim(x \wedge \bot) \to \sim\bot$

1. $\sim(\alpha \wedge \bot)$	Hypothesis
2. $\alpha \to \alpha$	Proposition 1
3. $\top$	Definition, 2
4. $\sim\bot$	Definition, 3
5. $\sim(\alpha \wedge \bot) \to \sim\bot$	DT, 1–4

4. $\sim\bot \to \sim(x \wedge \bot)$

1. $\sim\bot \to \sim(\bot \wedge \alpha)$	Ax21
2. $\sim(\bot \wedge \alpha) \to \sim(\alpha \wedge \bot)$	Ax22
3. $\sim\bot \to \sim(\alpha \wedge \bot)$	Lemma 4, 1, 2

For proving **QNSc.1**, $\sim\sim\sim x = \sim x$, we have:

1. $\sim\sim\sim\alpha \to \sim\alpha$, this is **Ax3**.

2. $\sim\alpha \to \sim\sim\sim\alpha$, this is instantiation of **Ax5**.

3. $\sim\sim\sim\sim\alpha \to \sim\sim\alpha$, this is instantiation of **Ax3**.

4. $\sim\sim\alpha \to \sim\sim\sim\sim\alpha$, this is instantiation of **Ax5**.

For proving **QNSc.2**, $\sim\sim(x \wedge y) = \sim\sim x \wedge \sim\sim y$, we have:

1. $\sim\sim(\alpha \wedge \beta) \to (\sim\sim\alpha \wedge \sim\sim\beta)$, this is **Ax27** ($\to$).

2. $(\sim\sim\alpha \wedge \sim\sim\beta) \to \sim\sim(\alpha \wedge \beta)$, this is **Ax27** ($\leftarrow$).

For proving **QNSc.3**, $\sim\sim x \wedge (y \oplus z) = (x \wedge y) \oplus (x \wedge z)$, we have:

1. $\sim\sim\alpha \wedge (\beta \oplus \gamma) \to (\alpha \wedge \gamma) \oplus (\alpha \wedge \gamma)$, this is **Ax28** ($\to$).

2. $((\alpha \wedge \gamma) \oplus (\alpha \wedge \gamma)) \to \sim\sim\alpha \wedge (\beta \oplus \gamma)$, this is **Ax28** ($\leftarrow$).

For proving **QNSc.4**, $x \oplus y \equiv x^2 \oplus y^2$, we have:

1. $(\alpha \oplus \beta) \to \alpha^2 \oplus \beta^2$, this is **Ax29** $(\to)$.
2. $(\alpha^2 \oplus \beta^2) \to (\alpha \oplus \beta)$, this is **Ax29** $(\leftarrow)$.

□

Proposition 13. $\mathcal{V}_{\mathbf{QNS}} \subseteq \mathtt{Alg}^*(\mathcal{L}_{\mathbf{QNS}})$.

Proof. Let $\mathbf{A} \in \mathbf{QNS}$, and let a, b, $c \in A$ be generic elements. By Theorem 14, we assume that $\mathbf{A}$ is a twist-structure, and from now on we also denote $a = \langle a_1, a_2\rangle$, $b = \langle b_1, b_2\rangle$ and $c = \langle c_1, c_2\rangle$. In the case of $E(a \leftrightarrow b)$ saying this is equivalent to proving that $\pi_1(a) = \pi_1(b)$, this is, $a_1 = b_1$.

The axioms **Ax1-Ax17** are also present in $\mathcal{L}_{\mathbf{QNP}}$, so their checks will be omitted. Furthermore, the axioms **Ax18-Ax27** are also present in $\mathcal{L}_{\mathbf{QN4}}$, so their checks will be omitted. For equations and quasi-equations of $\mathtt{Alg}^*(\mathcal{L}_{\mathbf{QNS}})$, we have

- $E((\sim\sim a \wedge (b \oplus c)) \leftrightarrow ((a \wedge b) \oplus (a \wedge c)))$

  On the one hand, $\pi_1[\sim\sim a \wedge (b \oplus c)] = \pi_1[\sim\sim a \wedge (\sim(\sim b \wedge \sim c))] = \pi_1[\sim\sim\langle a_1, a_2\rangle \wedge (\sim(\sim\langle b_1, b_2\rangle \wedge \sim\langle c_1, c_2\rangle))] = \pi_1[\sim\langle a_2, \Box a_1\rangle \wedge (\sim(\langle b_2, \Box b_1\rangle \wedge \langle c_2, \Box c_1\rangle))] = \pi_1[\langle \Box a_1, \Box a_2\rangle \wedge (\sim\langle b_2 \wedge c_2, \Box b_1 \oplus \Box c_1\rangle)] = \pi_1[\langle \Box a_1, \Box a_2\rangle \wedge \langle \Box b_1 \oplus \Box c_1, \Box(b_2 \wedge c_2)\rangle] = \pi_1[\langle \Box a_1 \wedge (\Box b_1 \oplus \Box c_1), \Box a_2 \oplus \Box(b_2 \wedge c_2)\rangle] = \Box a_1 \wedge (\Box b_1 \oplus \Box c_1) = (\Box a_1 \wedge \Box b_1) \oplus (\Box a_1 \wedge \Box c_1) = \Box(a_1 \wedge b_1) \oplus \Box(a_1 \wedge c_1)$.

  On the other hand, $\pi_1[(a \wedge b) \oplus (a \wedge c)] = \pi_1[\sim(\sim(a \wedge b) \wedge \sim(a \wedge c))] = \pi_1[\sim(\sim(\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle) \wedge \sim(\langle a_1, a_2\rangle \wedge \langle c_1, c_2\rangle))] = \pi_1[\sim(\sim\langle a_1 \wedge b_1, a_2 \oplus b_2\rangle \wedge \sim\langle a_1 \wedge c_1, a_2 \oplus c_2\rangle)] = \pi_1[\sim(\langle a_2 \oplus b_2, \Box(a_1 \wedge b_1)\rangle \wedge \langle a_2 \oplus c_2, \Box(a_1 \wedge c_1)\rangle)] = \pi_1[\sim\langle (a_2 \oplus b_2) \wedge (a_2 \oplus c_2), \Box(a_1 \wedge b_1) \oplus \Box(a_1 \wedge c_1)\rangle] = \pi_1[\langle \Box(a_1 \wedge b_1) \oplus \Box(a_1 \wedge c_1), \Box((a_2 \oplus b_2) \wedge (a_2 \oplus c_2))\rangle] = \Box(a_1 \wedge b_1) \oplus \Box(a_1 \wedge c_1)$.

- $E((a \oplus b) \leftrightarrow (a^2 \oplus b^2))$

  On the one hand, $\pi_1[a \oplus b] = \pi_1[\sim(\sim a \wedge \sim b)] = \pi_1[\sim(\sim\langle a_1, a_2\rangle \wedge \sim\langle b_1, b_2\rangle)] = \pi_1[\sim(\langle a_2, \Box a_1\rangle \wedge \langle b_2, \Box b_1\rangle)] = \pi_1[\sim\langle a_2 \wedge b_2, \Box a_1 \oplus \Box b_1\rangle] = \pi_1[\langle \Box a_1 \oplus \Box b_1, \Box(a_2 \wedge$

$b_2)\rangle] = \Box a_1 \oplus \Box b_1$.

On the other hand, $\pi_1[(a^2 \oplus b^2)] = \pi_1[\sim(\sim a^2 \wedge \sim b^2)] = \pi_1[\sim(\sim(a * a) \wedge \sim(b * b))] = \pi_1[\sim(\sim(\langle a_1, a_2\rangle * \langle a_1, a_2\rangle) \wedge \sim(\langle b_1, b_2\rangle * \langle b_1, b_2\rangle))] = \pi_1[\sim(\sim\langle a_1 \wedge a_1, (a_1 \to a_2) \wedge (a_1 \to a_2)\rangle) \wedge \sim(\langle b_1 \wedge b_1, (b_1 \to b_2) \wedge (b_1 \to b_2)\rangle)] = \pi_1[\sim(\sim\langle a_1, a_1 \to a_2\rangle \wedge \sim\langle b_1, b_1 \to b_2\rangle)] = \pi_1[\sim(\langle a_1 \to a_2, \Box a_1\rangle \wedge \langle b_1 \to b_2, \Box b_1\rangle)] = \pi_1[\sim\langle (a_1 \to a_2) \wedge (b_1 \to b_2), \Box a_1 \oplus \Box b_1\rangle] = \pi_1[\langle \Box a_1 \oplus \Box b_1, \Box((a_1 \to a_2) \wedge (b_1 \to b_2))\rangle] = \Box a_1 \oplus \Box b_1$.

We have to prove that $E(a \to a)$ and $E(\sim a \to \sim a)$. These are easy to check.

We have to prove that if $E(a)$ and $E(a \to b)$ then $E(b)$. Therefore, we will use the fact that $|a| \to b = b$. Note that,

$$\begin{cases} |a| = |\langle a_1, a_2\rangle| = \langle a_1, a_2\rangle \to \langle a_1, a_2\rangle = \langle a_1 \to a_1, \Box a_1 \wedge a_2\rangle = \langle 1, \Box a_1 \wedge a_2\rangle \\ |a| \to b = \langle 1, \Box a_1 \wedge a_2\rangle \to \langle b_1, b_2\rangle = \langle 1 \to b_1, \Box 1 \wedge b_2\rangle = \langle b_1, b_2\rangle = b \end{cases}$$

Thus $(a \to a) \to b = b$, but we have that $a \to a = a$ and therefore $a \to b = b$, but as $a \to b = (a \to b) \to (a \to b)$ and $a \to b = b$, then $b = b \to b$ and this is what we wanted to prove.

We have to prove that if $E(\Delta(a, b))$ then $a = b$. So, $E(a \to b)$, $E(b \to a)$, $E(\sim a \to \sim b)$, $E(\sim b \to \sim a)$, we give us $a_1 \leq b_1$, $b_1 \leq a_1$, $a_2 \leq b_2$, $b_2 \leq a_2$, respectively. Therefore $a = b$. $\Box$

Corollary 3. The class of $\mathcal{V}_{\mathbf{QNS}}$ and the class of $\mathtt{Alg}^*(\mathcal{L}_{\mathbf{QNS}})$-algebras coincide.

6 Conclusion

The present study began by laying down the basic concepts and terminologies involving algebra, logic and their algebrization. Later, by studying Hilbert calculi and its algebraic semantics, we were able to provide new results about his algebraic properties, carried out mainly through the mathematical tool called twist-algebra representation.

The dissertation also aimed to provide a better understanding of quasi-Nelson logics, by presenting an equivalent algebraic semantics for the logics of some fragments of quasi-Nelson logic, namely: pocrims ($\mathcal{L}_{\mathbf{QNP}}$) and semihoops ($\mathcal{L}_{\mathbf{QNS}}$); in addition to the logic of quasi-N4-lattices ($\mathcal{L}_{\mathbf{QN4}}$), many of which had never been considered in the literature so far.

However, Busaniche et al. introduces in [7], a very general twist construction based on the notion of Nelson conucleus, whose main idea is that the various twist representations can be obtained uniformly by employing a unary function that realizes, in each algebra, a special interior operator (a conucleus). This approach is extended to quasi-Nelson algebras, which may suggest an applicability to the subreducts considered in this work.

As prospects for future work, we suggest the use of the Lean functional programming language, which can also be used as an interactive theorem prover, for the proofs involved in this research. Thus, the derivations necessary for the completeness proofs will be verified with the aid of this tool. In this way, the presentation of the proof given in this work will be more accessible, understandable and trustworthy to the community.

Research on quasi-N4-lattices is in a preliminary stage, and only time will tell to what extent further investigations on this and related classes of algebras will prove fruitful.

Refining the twist construction. By Theorem 10, we know that we can identify an arbitrary quasi-N4-lattice $\mathbf{A}$ with a subalgebra of $\mathbf{B}^{\bowtie}$ for some nuclear Brouwerian algebra $\mathbf{B}$. This establishes a correspondence (which may be rephrased as an adjunction between suitably defined categories) between each nuclear Brouwerian algebra $\mathbf{B}$ and the family of quasi-N4-lattices that canonically embed into $\mathbf{B}$. As shown in [22, Prop. 2.5], two further parameters ∇ and Δ (respectively, a lattice filter and an ideal of $\mathbf{B}$) are sufficient to uniquely determine a twist-algebra having the following set as underlying universe:

$$Tw(\mathbf{B}, \nabla, \Delta) := \{\langle a_1, a_2 \rangle \in B \times B : a_2 = \Box a_2,\ a_1 \vee a_2 \in \nabla,\ a_1 \wedge a_2 \in \Delta\}.$$

We thus have a one-to-one correspondence between triples $(\mathbf{B}, \nabla, \Delta)$ and quasi-N4-lattices, but we do not currently know whether *every* quasi-N4-lattice arises in this way. If the latter was true, then the correspondence would yield an equivalence between the algebraic category of quasi-N4-lattices and a category having as objects triples $(\mathbf{B}, \nabla, \Delta)$; this is indeed known to hold for N4-lattices [24].

Quasi-N4-lattices and relevant algebras. The paper [12] introduced the variety of *generalized Sugihara monoids* as a non-involutive generalization of algebraic models of the relevant logic *R-mingle*, a class of algebras known as *Sugihara monoids*. One of the main results of Galatos and Raftery is that generalized Sugihara monoids are representable through a twist construction which has striking similarities with the one for quasi-N4-lattices. The factor algebras employed in their twist construction are in fact nuclear Brouwerian algebras that are also prelinear (i.e. representable as subdirect products of linearly ordered ones).

While the equational properties of the two above-mentioned classes of algebras suggest that a direct comparison between (generalized) Sugihara monoids and (quasi-) N4-lattices is not likely to prove fruitful, we speculate that the twist construction may be used to establish a meaningful connection. Indeed, since the twist representation is used in [12] to establish a categorical equivalence between generalized Sugihara monoids and prelinear nuclear Brouwerian algebras, it may be possible to apply a similar strategy to quasi-N4-lattices, namely, single out a subcategory of (perhaps enriched) quasi-N4-lattices that may be proved to be equivalent as a category to the prelinear nuclear Brouwerian algebras considered in [12]. An equivalence with generalized Sugihara monoids would then be obtained as an immediate corollary.

Connexive Algebras. Heinrich Wansing in 2005, [27], introduces the Connexive Logic $\mathcal{C}$, and his presentation suggests that $\mathcal{C}$ is a constructive logic; thus, related to David Nelson's constructive logic with strong negation. $\mathcal{C}$ is the logic determined by the Hilbert-style calculus having modus ponens as its only rule, and the schematic axioms: (**C1**) the axioms of Positive Intuitionistic Logic, (**C2**) $\sim\sim\alpha \leftrightarrow \alpha$, (**C3**) $\sim(\alpha \vee \beta) \leftrightarrow \sim(\sim\alpha \wedge \sim\beta)$, (**C4**) $\sim(\alpha \wedge \beta) \leftrightarrow \sim(\sim\alpha \vee \sim\beta)$, (**C5**) $\sim(\alpha \to \beta) \leftrightarrow (\alpha \to \sim\beta)$. Recently, Fazio and Odintsov, in [9], show that axiomatic extensions of $\mathcal{C}$ are BP-algebraizable with respect to varieties of **C**-algebras, using twist-products (specifically, full connexive twist structure). It is worthwhile investigating the relationship between the representation by Fazio and Odintsov ([9]) of **C**-algebras and the twist representation of N4-lattices, and possible generalizations to a non-involutive setting.

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